SPECTRAL PROPERTIES OF SCHRÖDINGER OPERATORS DEFINED ON N-DIMENSIONAL INFINITE TREES

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ABSTRACT. We study the discreteness of the spectrum of Schrödinger operators which are defined on N-dimensional rooted trees of a finite or infinite volume, and are subject to a certain mixed boundary condition. We present a method to estimate their eigenvalues using operators on a one-dimensional tree. These operators are called *width-weighted operators*, since their coefficients depend on the section width or area of the N-dimensional tree. We show that the spectrum of the width-weighted operator tends to the spectrum of a one-dimensional limit operator as the sections width tends to zero. Moreover, the projections to the one-dimensional tree of eigenfunctions of the N-dimensional Laplace operator converge to the corresponding eigenfunctions of the one-dimensional limit operator.

CONTENTS

1. INTRODUCTION

Let T_1 be a one-dimensional infinite tree. We assume throughout this paper that T_1 is regular (see Definition 2.1 and Remark 1.1). For $N \geq 2$, we also consider an ε-inflated tree T_N^ε around T_1 which is an N-dimensional offset (or

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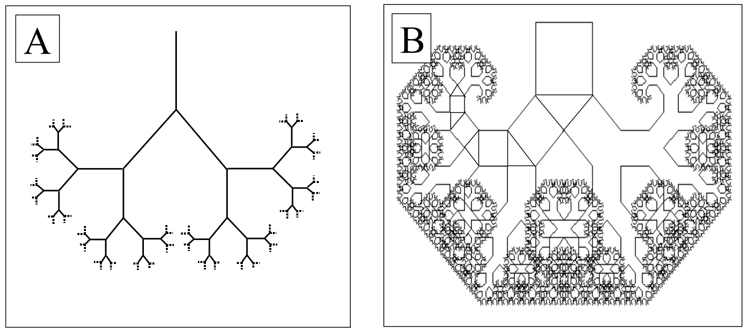

FIGURE 1. An example of one and two dimensional trees. **A.** One-dimensional tree. **B.** Two-dimensional tree presented in $\mathbb{R}^2$. Some of its triangle connectors and rectangle edges are emphasized.

fattening) of T_1. See Figure 1 for an illustration of a 2-dimensional tree $T_2 = T_2^1$.

We prove ε-dependent estimates for the spectrum of the eigenvalue problem

$$L_\varepsilon u := -\Delta u + W_{T_N^\varepsilon} u = \lambda^\varepsilon u \qquad \text{on } H_0^1(T_N^\varepsilon), \tag{1.1}$$

subject to the Neumann boundary condition on ∂T_N^ε except on the root of the tree, where we impose the Dirichlet boundary condition. We assume that $W_{T_N^\varepsilon}$ is a bounded and continuous potential on T_N^ε. Specifically, we show that if T_N^ε has a finite radius, then under some further assumptions the spectrum of L_ε is discrete and the eigenvalues of the Schrödinger operators L_ε satisfy $\lambda_i^\varepsilon \to \mu_i$ as $\varepsilon \to 0$, where μ_i are the eigenvalues of the following *weighted* Schrödinger operator on T_1

$$\overline{L}u := -\frac{1}{\rho}(\rho u')' + W_{T_1} u = \mu_i u. \tag{1.2}$$

Here $\rho > 0$ is a weight function on T_1 defined in terms of the inflation T_N^ε, and W_{T_1} is the cross section average of $W_{T_N^\varepsilon}$.

The spectral behavior of the Neumann Laplacian and Schrödinger operators on thin domains has been extensively investigated. Indeed, in [21], Rubinstein and Schatzman study the relation between the spectral properties of the Laplace operator defined on a metric *graph* G and on a strip shaped domain G^ε of width ε around G. The results of [21] on the spectrum of the Laplacian cannot be applied to our trees because of the following essential differences between the problems:

(1) Rubinstein and Schatzman treat the case in which the graph G has a *finite* number of vertices, while our tree T_1 has an infinite number of vertices.
(2) They consider graph-surrounding domains having a constant (uniform) width. In the case of an infinite trees, the discreteness of the spectrum imposes that the width of higher branches of the tree must be scaled.
(3) In particular, the inflated finite graph is of finite volume, while our inflated infinite tree may have infinite volume.

In [10], Kuchment and Zeng extend the results in [21]. For example, the conditions on the smoothness of the boundary of the domain near the vertices were relaxed and the constant width of the surrounding domain is not assumed.

Since T_1 in our case is an infinite tree, the results of [10, 21] do not apply in our setting. Nevertheless, we were able to modify the approach in [21] to obtain similar results in the infinite case. In particular, we could not compare directly

the eigenvalue λ_i^ε to μ_i. Instead, we find it more convenient to compare the spectra of $-\Delta + W_{T_N^\varepsilon}$ on T_N^ε to the Schrödinger operator on T_1 subjected to a pair of $\varepsilon-$dependent weight functions $\rho_{1,\varepsilon}$, $\rho_{2,\varepsilon}$, satisfying $\rho_{1,\varepsilon}, \rho_{2,\varepsilon} \to \rho$ as $\varepsilon \to 0$. So, we replace (1.2) by

$$\overline{L}_\varepsilon u := -\frac{1}{\rho_{2,\varepsilon}}(\rho_{1,\varepsilon} u')' + W_{T_1} u = \mu_i^\varepsilon u\,,$$

and prove that the λ_j^ε is approximated, on the one hand, by μ_j^ε while the later is approximated by μ_j for ε small.

Spectral properties of Schrödinger operators defined on infinite one-dimensional metric trees and graphs has also been intensively studied. In [4], Carlson shows that if G is a connected metric graph which has a finite total edges length (a finite volume), then the Laplacian defined on G has a compact resolvent and therefore a discrete spectrum. Solomyak and Naimark have developed general tools for studying spectral properties of Schrödinger operators on metric graphs and trees (see, for example, [14, 15, 22, 23]). In [22], Solomyak has proved that if T_1 is a regular tree whose radius is finite, and if $W_{T_1}(x)$ is a radial measurable real valued function which is bounded below, then the spectrum of $\overline{L}$ is discrete.

Solomyak's result is stated for trees of uniform weight function ρ and its proof relies on the monotonicity of g, where $g(t)$ is the number of branches which contain points of distance t from the root. In fact, to adjust Solomyak's proof for our case, one needs to assume only that $g\rho$ is a monotone nondecreasing. If ρ is constant then it is a natural assumption, but if $\rho(t)$ is decreasing (as in our case), this monotonicity may be violated. So, we extend this result under a milder condition on $g\rho$.

We prove the discreteness of the spectrum of Schrödinger operators on regular N-dimensional trees with infinite volume, as long as the tree radius is finite. Our proof relies on a lemma of Lewis [11, Lemma 1]. The proof of the discreteness in the N-dimensional case can be applied also to show that the L^2-norm of functions which are bounded in $H_0^1(T_N^\varepsilon)$ does not accumulate at the tree connectors or ends.

A natural question emerging from the correspondence between the eigenvalues of N-dimensional Laplace operator, and one-dimensional width-weighted operators, is whether the corresponding *eigenfunctions* present the same convergence behavior. In [7, 8], Kosugi has proved that solutions of (semilinear) elliptic equations on finite N-dimensional trees indeed converge as the width tends to zero to solutions of width-weighted equations. We present a different method and prove that certain projections of eigenfunctions of the Laplace operator on T_N^ε converge to the corresponding eigenfunctions on T_1. In contrast to [7, 8], we treat infinite trees rather than trees with a finite number of vertices. In addition, our assumptions on the smoothness of the connectors are much weaker than those in [7, 8], and in fact, we require only that the connectors have a Lipschitz boundary.

Remark 1.1. Our method applies to more general setting. But to facilitate the presentation, we restrict our study in the present paper to the case where T_1 is a regular metric tree, and the inflated N-dimensional tree is a self-similar radial tree with 'cylindrical' edges.

We wish to mention two more articles which study the spectrum of thin domains. In an earlier article [9], Kuchment and Zeng study the dependence of the spectrum of the Neumann Laplacian on the behavior of the surrounding thin domain near the vertices. They found differential operators on the graph which

correspond to the case in which the neighborhoods of the vertices are much larger or much smaller than the tubes connecting them. In [5], Evans and Saito proved results about the connection between the essential spectrum of the Neumann Laplacian on thin domains surrounding trees and the essential spectrum of their skeletons. They apply their results on horns, spirals, "rooms and passages" domains and domains with fractal boundaries. In our case the essential spectrum is empty, as was mentioned above.

The motivation for our problem is that fractal structures, and in particular, fractal tree-like structures, have a vast applications range. For example, fractal geometry is used in order to form antennas, which present a multi-band behavior (see [1, 18]). In [19], Puente et al. state that fractal tree shaped antennas have a denser band distribution than previously reported Sierpinski fractal antennas. Estimating the eigenvalues of the Laplace operator defined on such domains may help in specifying the natural transmission frequencies for the antennas.

Another applications field for fractal geometry is medical modelling. Nelson et al. mention in [16] that fractal models can be applied to human lungs, vascular tree, neural networks, urinary ducts, brain folds and cardiac conduction fibers. Fractal models of human lungs can be found also in [12, 17, 24].

The outline of this article is as follows. In Section 2, we present the basic notations we use, describe the class of trees we are interested in, and define the operators on the trees. Section 3 is devoted to the study of the behavior of H^1-functions near the vertices. In Section 4, we prove the discreteness of the spectrum of Schrödinger operators on T_1 and T_N. The convergence (as $\varepsilon \to 0$) of the spectrum of $\{\overline{L}_\varepsilon\}$, the operator sequence defined on T_1, to the spectrum of the limit operator $\overline{L}$ is proved in Section 5.

In sections 6.1.1 and 6.1.2 we define transformations between $H^1_0(T_N)$ and $H^1_{0,\rho_2}(T_1)$ and prove comparison theorems for the Rayleigh quotients of the one and N-dimensional operators. In Section 6.2, we use these comparison theorems to characterize the behavior of the spectrum on T_N. Finally, the convergence of projections of N-dimensional eigenfunctions of Laplace operator to eigenfunctions of the one-dimensional width-weighted operators is proved in Section 7.

2. Preliminaries

2.1. General notations.

(1) Throughout the article, $c, c_1, c_2, \ldots$, and C denote constants, whose exact values are irrelevant, and may change from line to line.
(2) Let $\{a_j\}$ and $\{b_j\}$ be positive sequences. We denote $a_j \asymp b_j$ if there exists a constant $c > 0$ such that $c^{-1} \leq a_j/b_j \leq c$ for all $j \in \mathbb{N}$. We use a similar notation for positive functions, i.e., we denote $f \asymp g$ if there exists a constant $c > 0$ such that $c^{-1} \leq f(x)/g(x) \leq c$ for all x in the domain of the functions f and g.
(3) For a domain $\Omega \subset \mathbb{R}^N$, we denote by $|\Omega|$ its volume in $\mathbb{R}^N$.

2.2. The tree T_1.

(1) T_1 is a one-dimensional connected rooted metric tree. It contains an infinite number of vertices v, connected by edges e.
(2) The root O of T_1 is a distinguished (and unique) vertex. Its generation number is defined to be zero.

(3) A vertex of T_1 is of generation j if it is connected to the root by a succession of j edges. The generation of a given vertex v is denoted by $\mathrm{gen}(v)$.
(4) Likewise, e is an edge of generation j if it connect a pair of vertices of generations j and $j+1$, respectively. The generation number of a given edge e is denoted by $\mathrm{gen}(e)$.
(5) The Euclidian length of an edge e is denoted by $|e|$.
(6) The degree of a vertex v is $k(v)$. It is the number of edges connecting v to the vertices of generation $\mathrm{gen}(v)+1$.
(7) The set of all edges meeting at a vertex v is $N(v)$. There are exactly $k(v)+1$ edges in $N(v)$.
(8) The distance $\mathrm{dist}(x,y)$ between $x,y \in T_1$ is the Euclidian length of the path on T_1 connecting x to y. We denote $|x| := \mathrm{dist}\{O,x\}$.
(9) $g(t)$ is the counting function of T_1, namely, $g(t)$ is the number of edges which contain a point $x \in T_1$ with $|x| = t$.
(10) $R(T_1) \equiv \sup_{x\in T_1} |x|$ is the *radius* of T_1. $L(T_1) \equiv \sum_{e\in T_1} |e|$ is the *length* of T_1.

Definition 2.1. T_1 is called *radial* if the length $|e|$ of each edge e and the degree $k(v)$ of each vertex v depend only on $\mathrm{gen}(e)$ and $\mathrm{gen}(v)$, respectively. A radial tree is called *regular* if $k(v) = k$ is a constant, independent of the generation.

2.3. **The ε-inflated N-dimensional tree.** The tree T_1 defined above is, in fact, a *combinatorial object*, but we always treat it as a metric tree or quantum graph. We shall now describe a way to construct an N-dimensional manifold which is an ε-inflation of T_1. For simplicity we shall assume that T_1 is radial and regular.

(1) A Lipschitz domain $\Omega \subset \mathbb{R}^{N-1}$ is given. It corresponds to the (scaled) *cross section* of the edges. We take the origin of $\mathbb{R}^{N-1}$ to be an interior point of Ω, called the *center* of Ω.
(2) A Lipschitz domain $V \subset \mathbb{R}^N$ is given. It corresponds to the (inflated) *vertices*. We take the origin of $\mathbb{R}^N$ to be an interior point of V, called the *center* of V.
(3) $0 < \delta < 1$ is the scaling factor. The notation $\delta\Omega$ stands for the scaled domain $\delta\Omega := \{\delta x \mid x \in \Omega\}$. Similarly $\delta V := \{\delta x \mid x \in V\}$.
(4) The boundary of V contains $k+1$ disjoint sections: One of these sections is an isometric image of Ω, denoted by S_0. The other k sections are isometric images of $\delta\Omega$, and denoted by $S_1, \dots S_k$.
(5) The orthogonal projections of the center of V into S_0 and $S_j \subset \partial V$ for $1 \le j \le k$ coincide with the isometric image of the centers of Ω and $\delta\Omega$, respectively.

Next, we define the *inflated tree* T_N^ε. For this, let us consider a certain embedding of T_1 in $\mathbb{R}^{N+1}$. We denote this embedding of T_1 by the same name, T_1. It is, in fact, determined by the choice of the inflated vertex V, to be explained below:

(6) For each vertex v in the embedded tree T_1, the inflated vertex is an isometric image of $V^\varepsilon(v) := \varepsilon\delta^{\mathrm{gen}(v)}V$ whose center coincides with v.
(7) Each edge $e \in N(v)$ is perpendicular to $S_e^\varepsilon(v)$, where $S_e^\varepsilon(v)$ is the isometric image of the section of $\partial V^\varepsilon(v)$ intersecting the edge e.
(8) The *skeleton* of $V^\varepsilon(v)$ is $\overline{V}^\varepsilon(v) := V^\varepsilon(v) \cap T_1$.
(9) For each edge e of the embedded T_1, the inflated edge is
$$E^\varepsilon(e) = e \times S_e^\varepsilon(v) \setminus \cup_v V^\varepsilon(v) \ .$$
(10) The *skeleton* of $E^\varepsilon(e)$ is $\overline{E}^\varepsilon(e) := E^\varepsilon(e) \cap T_1$.

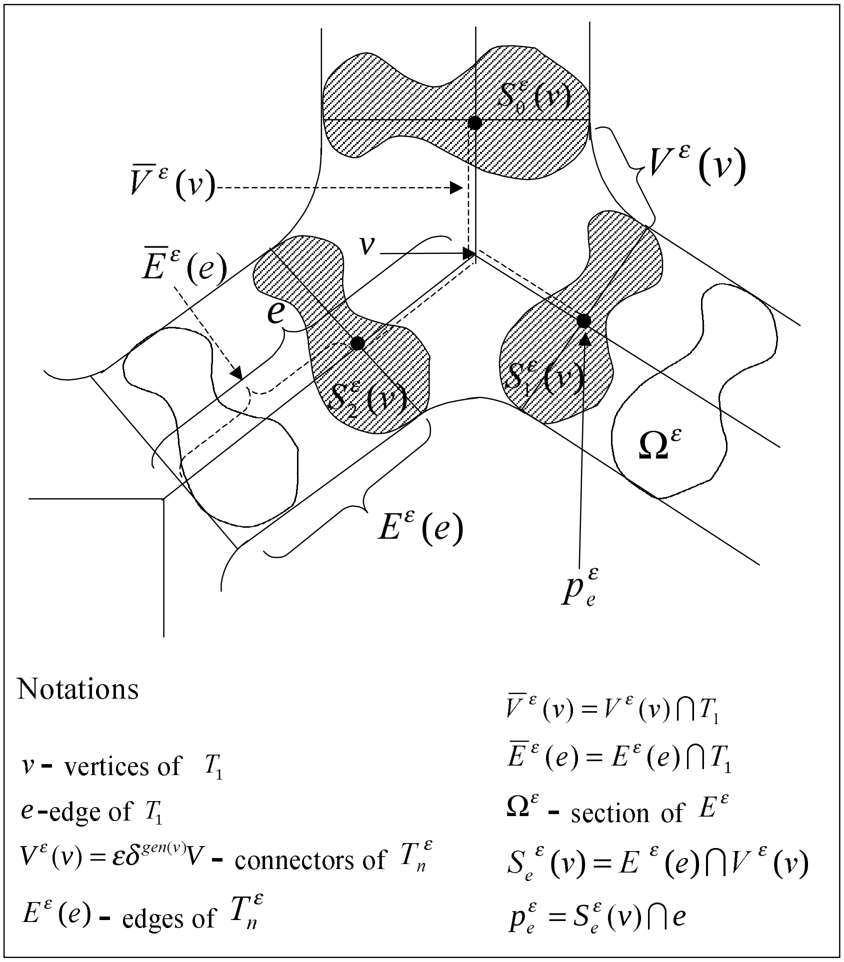

FIGURE 2. Notations of parts of T_1 and T_n^ε.

An inflated 2-dimensional tree is depicted in Figure 2. A somewhat degenerate example of an inflated tree is the *straightened tree*, which we denote by $\hat{T}_N$. We use $\hat{T}_N$ as a canonical representation for T_N in Section 4.2.

Definition 2.2 (The straightened tree)**.** The inflated vertex $\hat{V}$ is given by the cylinder $\hat{\Omega} \times [0, -1]$. The section $\hat{S}_0 := \hat{\Omega} \times \{0\}$ is the top of $\hat{V}$, and its base $\hat{\Omega} \times \{-1\}$ consists of k disjoint isometric copies of $(k)^{-1/N}\hat{\Omega} \times \{-1\}$, corresponding to the sections $\hat{S}_1, \dots \hat{S}_k$. A two-dimensional straightened tree is depicted in Figure 3. The above condition implies that $\hat{\Omega}$ is a box in $\mathbb{R}^N$ of a certain type which depend on k and N. Indeed, take a box $\hat{\Omega}$ whose sizes are $(1, k^{1/N}, k^{2/N}, \dots, k^{(N-1)/N})$, then k copies of $(k)^{-1/N}\hat{\Omega}$ exactly cover $\hat{\Omega}$. We may of course consider also other tilings.

Corollary 2.3. *The straightened tree $\hat{T}_N$ can be parameterized by the cylinder $\hat{\Omega} \times \hat{R}$, where $\hat{R}$ is its radius (see Figure 3).*

2.4. **Cross sections and functions on T_1 and T_N^ε.** There is a natural coordinate system on each of the edges $E^\varepsilon(e) \subset T_N^\varepsilon$, namely $\vec{x} \in E^\varepsilon(e) \subset T_N^\varepsilon$ is parameterized as $\vec{x} = \mathbf{x} = (\vec{s}, \theta)$, where $\vec{s}$ is a parameterization of the corresponding perpendicular section S_e in Ω, scaled by $\varepsilon\delta^{\text{gen}(e)}$, and θ is a parameterization of $e \setminus \cup_v \overline{V}^\varepsilon(v)$. We can also use the natural parameterization of V, scaled by $\varepsilon\delta^{\text{gen}(v)}$,

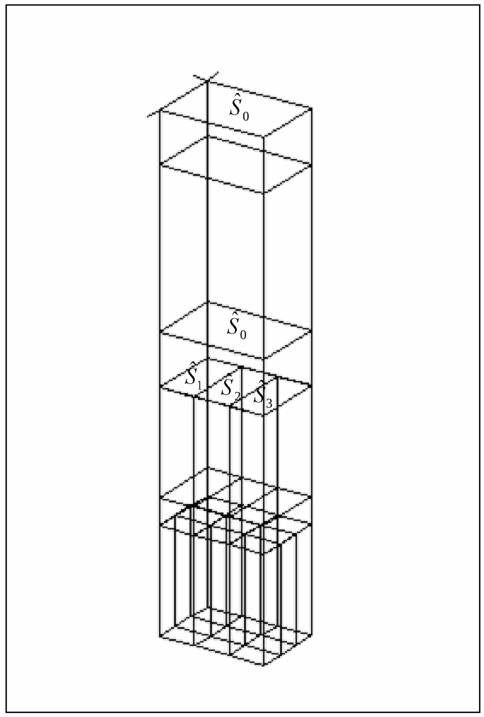

FIGURE 3. The straightened tree, $\hat{T}_3$ for $k = 3, N = 3$.

to describe the coordinate system in the inflated vertex $V^\varepsilon(v)$. We always take the center of $V^\varepsilon(v)$ as the origin $0 \in V$.

(1) We denote by f_e the restriction of a function f on T_1 to an edge e. In most cases we omit this notation and write simply f instead of f_e.
(2) The function ρ^* is defined on T_1 by $\rho^*_e = \delta^{(N-1)\mathrm{gen}(e)}|\Omega|$.
(3) Let f be a function on T^1_N. We denote by f^ε the following rescaling of f on T^ε_N :

$$f^\varepsilon(\vec{x}) = f_{T^\varepsilon_N}(\vec{x}) := \begin{cases} f(\theta, \vec{s}/\varepsilon) & \vec{x} = (\theta, \vec{s}) \in E^\varepsilon(e), \\ f(\vec{x}/\varepsilon) & \vec{x} \in V^\varepsilon(v), \end{cases}$$

(4) The total cross section of T_N is defined for $t \in T_1 \subset T_N$ as $H(t) = g(t)\rho^*(t)$, where g is the counting function of the skeleton T_1 of T_N and ρ^* as defined in (2) above.

2.5. **Function spaces.**

(1) Let $\rho > 0$ be a measurable (weight) function on T_1. Denote

$$L_{2,\rho}(T_1) = \left\{ f \mid f \text{ is measurable on } T_1 \text{ and } \int_{T_1} |f|^2 \rho \, \mathrm{d}\theta < \infty \right\}.$$

The space $L_{2,\rho}(T_1)$ equipped with the inner-product $<f, g>_\rho := \int_{T_1} f \overline{g} \rho \, \mathrm{d}\theta$ is a Hilbert space.
(2) $C^1(T_1)$ is the space of continuous functions f on T_1, such that $f_e \in C^1(e)$ for each edge e. Let $\rho > 0$ be a measurable function on T_1. $H^1_\rho(T_1)$ is the completion of the space

$$\left\{ f \in C^1(T_1) \mid \sum_{e \in T_1} \int_e \left(|(f_e)'|^2 + |f_e|^2 \right) \rho_e \, \mathrm{d}\theta < \infty \right\}$$

with respect to the norm $||f||_{H^1_\rho(T_1)} := \left[\sum_{e\in T_1}\int_e \left(|f_e'|^2+|f_e|^2\right)\rho_e \,\mathrm{d}\theta\right]^{1/2}$.

(3) $H^1_{0,\rho}(T_1)$ is the completion in $H^1_\rho(T_1)$ of $C^1_0(T_1)$. For the weight function ρ^*, we abbreviate $H^1_{0,*}(T_1) := H^1_{0,\rho^*}(T_1)$.

(4) $H^1(T_N)$ is the completion of the space

$$\left\{ f \in C^1(T_N) \mid \int_{T_N} \left(|\nabla f|^2 + |f|^2\right) \mathrm{d}\mathbf{x} < \infty \right\}$$

with respect to the norm $||f||_{H^1(T_N)} := \left[\int_{T_N} \left(|\nabla f|^2+|f|^2\right)\mathrm{d}\mathbf{x}\right]^{1/2}$.

(5) $H^1_0(T_N)$ is the completion in $H^1(T_N)$ of all functions in $C^1(T_N)$ satisfying $f|_{O\times\Omega_0} = 0$.

2.6. **Laplace and Schrödinger operators on T_1 and T_N.** We define a family of operators on T_1 using the standard definition of operators on T_1 (see [21, 22]).

Let $W \in L^\infty(T_1)$ be a bounded real valued potential, and let ρ_α and ρ_β be positive bounded $L^1_{\mathrm{loc}}(T_1)$ weight functions, which satisfy $\rho_\alpha \asymp \rho_\beta$. In particular, $H^1_{0,\rho_\alpha}(T_1)$ and $H^1_{0,\rho_\beta}(T_1)$ are equivalent in the sense that $u \in H^1_{0,\rho_\alpha}(T_1)$ if and only if $u \in H^1_{0,\rho_\beta}(T_1)$, and there exists a constant $c > 0$ independent of u such that

$$\frac{1}{c}||u||_{H^1_{0,\rho_\alpha}(T_1)} \le ||u||_{H^1_{0,\rho_\beta}(T_1)} \le c||u||_{H^1_{0,\rho_\alpha}(T_1)}.$$

We denote by

$$E(u,v) := \sum_{e\in T_1}\int_e \left[\frac{\rho_\alpha}{\rho_\beta}(u_e)'(\overline{v}_e)' + W u_e \overline{v}_e\right]\rho_\beta \,\mathrm{d}t$$

the bilinear form on $H^1_{0,\rho_\beta}(T_1) \times H^1_{0,\rho_\beta}(T_1)$. Without loss of generality, we may assume that $E \ge 0$ on $C^1_0(T_1)$, so, E is a symmetric and nonnegative closed bilinear form, and $H^1_{0,\rho_\beta}(T_1)$ is dense in $L^2_{\rho_\beta}(T_1)$. By Friedrich's extension theorem (see e.g. Theorem X.23 in [20]) or the First Representation Theorem (see Theorem VI.2.1 in [6]), there exists a unique selfadjoint operator $L_{\alpha,\beta}$ such that $\mathrm{Dom}(L_{\alpha,\beta}) \subseteq \mathrm{Dom}(E)$ and $E(u,v) =< L_{\alpha,\beta}u, v >_{\rho_\beta}$ for all $u \in \mathrm{Dom}(L_{\alpha,\beta})$ and $v \in H^1_{0,\rho_\beta}(T_1)$. By this theorem, the domain of $L_{\alpha,\beta}$ is given by:

$$\mathrm{Dom}(L_{\alpha,\beta}) = \{u \in H^1_{0,\rho_\beta}(T_1) \mid \ |E(u,v)| \le C|v|_{L^2_{\rho_\beta}(T_1)} \ \forall v \in H^1_{0,\rho_\beta}(T_1)\}$$

for some constant C. Moreover, it is well known (see e.g. [21]) that the domain of $L_{\alpha,\beta}$ is contained in the space of all functions u satisfying the following *Kirchhoff conditions*:

(1) u is continuous at the vertices (since $H^1_{0,\rho_\beta} \subset C(T_1)$).

(2) $\sum_{e\in N(v)}(\rho_\alpha)_e(u_e(v))' = 0$ in each vertex $v \in T_1$.

We will call operators of this form *width-weighted operators*, because we will use them for weights ρ_α and ρ_β which are closely related to the width or section area of T_N. Similar operators are also presented by Evans and Saito in [5].

Remark 2.4. The domain of the operator $L_{\alpha,\beta}$ is clearly dense in $H^1_{0,\rho}$ for $\rho = \rho_\alpha$ or $\rho = \rho_\beta$.

Finally, the Laplace operator on the tree T_N is defined by the Friedreich extension of the quadratic form

$$E_N(u,w) := \int_{T_N} \nabla u \cdot \nabla \bar{w} \,\mathrm{d}\mathbf{x} \tag{2.1}$$

for u, w in the space $H^1_0(T_N)$ (see the definition of $H^1_0(T_N)$ in Section 2.5 §(5)).

3. Behavior of functions near the vertices

Here we concentrate on a neighborhood of a vertex (resp. an inflated vertex) in T_1 (resp. T_N^ε). For T_1, we shall consider the skeleton $\overline{V}^\varepsilon(v)$ corresponding to a vertex v, as defined in Section 2.3 §(8). We shall also denote the "canonical" skeleton, corresponding to $\varepsilon = 1$, by $\overline{V}(v)$. Occasionally, we shall omit the reference to a particular vertex v and just denote it as $\overline{V}$. The end points of $\overline{V}(v)$ are denoted by p_e, where $e \in N(v)$ (see Figure 2). Recall that ρ^*, as defined in Section 2.4 §(2), is a positive weight function on T_1, which is constant on each edge.

(1) For each edge $e \in N(v)$ define a nonnegative function $\psi_{(e)} \in C^1(\overline{V})$ such that $\psi_{(e)}(p_e) = 1$ and $\psi_{(e)}(p_{\tilde{e}}) = 0$ for $\tilde{e} \neq e$. We also assume that

$$\sum_{e \in N(v)} \psi_{(e)} = 1 \quad \text{on} \quad \overline{V}(v)\ . \tag{3.1}$$

If the skeleton $\overline{V}$ is scaled by $\delta > 0$, so $\overline{V} \to \delta\overline{V} := \left\{\delta\theta \mid \theta \in \overline{V}\right\}$, where the vertex v is taken as the origin, then $\psi_{(e)}$ is scaled into $\psi^\delta_{(e)}(x) := \psi_{(e)}(x/\delta)$ for any $x \in \delta\overline{V}$.

(2) Let V be the "canonical" inflated vertex defined in Section 2.3 §(2). We choose a family of nonnegative functions $\phi_{(e)} \in C^1(V) \cap C(\bar{V})$ such that

$$\phi_{(e)}(\mathbf{x}) = \begin{cases} 1 & \theta(\mathbf{x}) \in S_e, \\ 0 & \theta(\mathbf{x}) \in S_{\tilde{e}},\ \text{where } \tilde{e} \neq e, \end{cases}$$

and

$$\sum_{e \in N(v)} \phi_{(e)} = 1 \qquad \text{on } V. \tag{3.2}$$

Similarly, if V is scaled by $\delta > 0$, so $V \to \delta V := \{\delta x \mid x \in V\}$, where the center of V is taken as the origin, then $\phi_{(e)}$ is scaled into $\phi^\delta_{(e)}(x) := \phi_{(e)}(x/\delta)$ for any $x \in \delta V$.

(3) Next, define for each $e \in N(v)$ the quadratic $(k+1) \times (k+1)$ matrices:

$$\overline{\mathbf{A}}_{l,m} := \int_{\overline{V}} (\psi_{(l)})'(\psi_{(m)})'\rho^*\, \mathrm{d}\theta, \quad \mathbf{A}_{l,m} := \int_V \nabla\phi_{(l)} \cdot \nabla\phi_{(m)}\, \mathrm{d}\mathbf{x},$$

and

$$\overline{\mathbf{B}}_{l,m} := \int_{\overline{V}} \psi_{(l)}\psi_{(m)}\rho^*\, \mathrm{d}\theta, \quad \mathbf{B}_{l,m} := \int_V \phi_{(l)}\phi_{(m)}\, \mathrm{d}\mathbf{x}.$$

(4) Let $\vec{1} := \left(1/\sqrt{k+1}, \ldots 1/\sqrt{k+1}\right) \in \mathbb{R}^{k+1}$, and for any $\vec{f} \in \mathbb{C}^{k+1}$ denote

$$\vec{f} \llcorner \vec{1} := \vec{f} - \left(\vec{f} \cdot \vec{1}\right)\vec{1}, \tag{3.3}$$

where $\cdot$ is the standard inner product in $\mathbb{C}^{k+1}$.

The following Lemma is elementary, but essential for our analysis.

Lemma 3.1. *The matrices* $\mathbf{A}$ *and* $\overline{\mathbf{A}}$ *are nonnegative definite, and* $\mathbf{B}$ *and* $\overline{\mathbf{B}}$ *are strictly positive definite. In particular, there exist constants* $\alpha^A > 0$, $\alpha^{\overline{A}} > 0$, $\alpha^B > 0$, *and* $\alpha^{\overline{B}} > 0$, *such that*

$$\frac{1}{\alpha^{\overline{A}}}|\vec{f}\llcorner\vec{1}|^2 \le \vec{f}\cdot\overline{\mathbf{A}}\vec{f}^* \le \alpha^{\overline{A}}|\vec{f}\llcorner\vec{1}|^2, \qquad \frac{1}{\alpha^{A}}|\vec{f}\llcorner\vec{1}|^2 \le \vec{f}\cdot\mathbf{A}\vec{f}^* \le \alpha^{A}|\vec{f}\llcorner\vec{1}|^2, \tag{3.4}$$

and

$$\frac{1}{\alpha^{\overline{B}}}|\vec{f}|^2 \le \vec{f}\cdot\overline{\mathbf{B}}\vec{f}^* \le \alpha^{\overline{B}}|\vec{f}|^2, \qquad \frac{1}{\alpha^{B}}|\vec{f}|^2 \le \vec{f}\cdot\mathbf{B}\vec{f}^* \le \alpha^{B}|\vec{f}|^2 \tag{3.5}$$

for all $\vec{f}\in\mathbb{C}^{k+1}$, *where* $\vec{f}^*$ *denotes the complex conjugate of* $\vec{f}^t$.

Proof. The non-negativity (resp. positivity) of $\mathbf{A}$ and $\overline{\mathbf{A}}$ (resp. $\mathbf{B}$ and $\overline{\mathbf{B}}$) follows from the corresponding definitions, while (3.4) follows from (3.1) and (3.2). □

Let us introduce the following functionals on $H^1(\overline{V})$:

$$\overline{I}^{\overline{V}}_\gamma[g] := \int_{\overline{V}} (|g'|^2 + \gamma|g|^2)\rho^*\,\mathrm{d}\theta \qquad \text{for} \quad \gamma = 0,1, \tag{3.6}$$

and for $\vec{f}\in\mathbb{C}^{k+1}$ let us denote:

$$\overline{\mathcal{A}}_{\overline{V},\vec{f}} = \{g\in H^1(\overline{V}) \mid g(p_e) = f_e, \quad e\in N(v)\ \}. \tag{3.7}$$

Lemma 3.2. *Using the notations* (3.6) *and* (3.7), *we have for* $\gamma = 0,1$ *that*

$$\overline{J}^{\overline{V}}_\gamma[\vec{f}] := \inf_{g\in\overline{\mathcal{A}}_{\overline{V},\vec{f}}} \overline{I}^{\overline{V}}_\gamma[g]$$

is attained by a unique function h, *which solves the Dirichlet problem*

$$-h'' + \gamma h = 0 \quad \text{in } \overline{V}\cap e, \quad h(p_e) = f_e \quad \forall e\in N(v), \tag{3.8}$$

and satisfies Kirchhoff's conditions

$$\sum_{e\in N(v)} \rho^*_e h'_e(v) = 0. \tag{3.9}$$

Proof. The existence of minimizers u for $\overline{I}^{\overline{V}}_0$ and $\overline{I}^{\overline{V}}_1$, which satisfy (3.8) is standard (see e.g. the proof in [5, Theorem 2, pp. 448–449]).

We need to prove that the minimizer u of $\overline{I}^{\overline{V}}_\gamma$ satisfies Kirchhoff's derivatives condition. To this end, let $v\in C^1_0(\overline{V})$ and $0\neq\epsilon\in\mathbb{R}$. Since u is a minimizer, $I_\gamma[u]\le I_\gamma[u+\epsilon w]$, and therefore,

$$\int_{\overline{V}} (u'w' + \gamma uw)\rho^*\,\mathrm{d}\theta = 0.$$

By elliptic regularity $u\in C^2(V\cap e)$; Moreover, u is continuous in $\overline{V}$. Recall that ρ^* is constant on each edge, therefore, $-u'' + \gamma u = 0$ on $\overline{V}\cap e$. Thus,

$$\begin{aligned} 0 &= \sum_{e\in N(v)} \int_{\overline{V}\cap e} (u'w' + \gamma uw)\rho^*\,\mathrm{d}\theta \\ &= \sum_{e\in N(v)} \rho^*_e (u_e)' w_e \Big|_{p_e}^{v_e} + \sum_{e\in N(v)} \int_{\overline{V}\cap e} (-u'' + \gamma u) w\rho^*\,\mathrm{d}\theta = w(v) \sum_{e\in N(v)} \rho^*_e u'_e(v). \end{aligned} \tag{3.10}$$

The uniqueness of the minimizers of $\overline{I}^{\overline{V}}_0$ and $\overline{I}^{\overline{V}}_1$ follows since both are minima of strictly convex functionals on the underlying domains. □

Lemma 3.3. *There exist $\beta^{\overline{A}} > 0$ and $\beta^{\overline{B}} > 0$ such that for all $\delta > 0$*

$$\overline{I}_0^{\delta\overline{V}}[\vec{f}] \geq \delta^{-1}\beta^{\overline{A}}|\vec{f}\llcorner\vec{1}|^2, \tag{3.11}$$

and

$$\overline{I}_1^{\delta\overline{V}}[\vec{f}] \geq \delta^{-1}\beta^{\overline{B}}(|\vec{f}\llcorner\vec{1}|^2 + \delta^2|\vec{f}|^2). \tag{3.12}$$

Proof. In the following, we use the notations introduced in Lemma 3.2, and in (3.6) and (3.7). Consider the case $\delta = 1$ first. Let $\{\vec{\sigma}_e\}$ be the standard basis vectors in $\mathbb{C}^{k+1}$, where $e \in N(v)$. Let $h_{(e)} \in H^1(\overline{V})$ be the unique minimizer of $\overline{J}_\gamma^{\overline{V}}[\sigma_e]$. By Lemma 3.2 it follows that

$$\overline{J}_\gamma^{\overline{V}}[\vec{f}] = \overline{I}_\gamma^{\overline{V}}\left[\sum_{e\in N(v)} f_e h_{(e)}\right] = \sum_{e,\tilde{e}\in N(v)} f_e f_{\tilde{e}} \int_{\overline{V}} \left[h_{(e)}h'_{(\tilde{e})} + \gamma h_{(e)}h_{(\tilde{e})}\right]\rho^*\,\mathrm{d}\theta,$$

where each $h_{(e)}$ satisfies

$$-h''_{(e)} + \gamma h_{(e)} = 0 \quad \text{in } \overline{V}, \quad h_{(e)}(p_e) = 1, \quad h_{(e)}(p_{\tilde{e}}) = 0 \ \ \forall \tilde{e} \neq e.$$

Let $\gamma = 0$. By Lemma 3.2, $\overline{J}_0[\vec{f}]$ is attained uniquely by the harmonic function h which solves the corresponding Dirichlet problem (and satisfies Kirchhoff's conditions). In particular, it depends only on $\vec{f}$ and the domain $\overline{V}$. Since each solution h satisfying $h(p_e) = f_e$ can be presented uniquely by $h = \sum_{e\in N(v)} f_e h_{(e)}$, it follows that $\overline{J}_0[f]$ is a bilinear form. Clearly, it is a nonnegative $k+1$ dimensional form whose kernel contains only constant multiplicities of $\vec{1}$ for which the unique solution of the Dirichlet problem is constant. Therefore, it is equivalent to all nonnegative forms with such a kernel, and in particular, to $|\vec{f}\llcorner\vec{1}|^2$.

The proof for the case $\gamma = 1$ is similar except for replacing the Laplace operator by the operator $-\mathrm{d}^2/\mathrm{d}\theta^2 + 1$ and $|\vec{f}\llcorner\vec{1}|^2$ by $|\vec{f}|^2$.

Now, if $\delta < 1$ and $\gamma = 0$ we observe that the harmonic minimizers $h_{(e)}$ are scaled into $h_{(e)}(\cdot/\delta)$, and

$$\int_{\delta\overline{V}} h'_{(e)}(x/\delta)h'_{(\tilde{e})}(x/\delta)\rho^*\,\mathrm{d}\theta = \delta^{-1}\int_{\overline{V}} h'_{(e)}h'_{(\tilde{e})}\rho^*\,\mathrm{d}\theta.$$

For $\delta < 1$ and $\gamma = 1$, we use similar scaling argument to obtain (3.12). □

We wish to prove now the analog of Lemma 3.3 for the N-dimensional case. Consider the following functionals for $\gamma = 0, 1$:

$$I_\gamma[g] := \int_V (|\nabla g|^2 + \gamma|g|^2)\,\mathrm{d}\mathbf{x}. \tag{3.13}$$

For all $h \in H^1(V)$ and $0 \leq j \leq k$ we denote the *average* of h on the section $S_j \subset \partial V$ by

$$P_j(h) := \frac{1}{|S_j|}\int_{S_j} h\,\mathrm{d}\mathbf{s} \tag{3.14}$$

(see Section 2.3 §(4)). For $\vec{F} \in \mathbb{C}^{k+1}$ we define

$$\mathcal{A}_{V,\vec{F}} := \left\{g \in H^1(V) \mid P_j(g) = F_j \quad \forall j = 0, ..., k\right\}. \tag{3.15}$$

Lemma 3.4. *Let $\vec{F} \in \mathbb{C}^{k+1}$. Using the above notations, we have for $\gamma = 0, 1$ that*

$$J_\gamma[\vec{F}] := \inf_{g \in \mathcal{A}_{V,\vec{F}}} I_\gamma[g]$$

is attained by a function h, which is the unique solution of the problem

$$-\Delta h + \gamma h = 0 \quad in \quad V, \qquad h \in \mathcal{A}_{V,\vec{F}}\,, \tag{3.16}$$

and satisfies weakly the mixed boundary conditions

$$\frac{\partial h}{\partial n} = 0 \quad on \quad \partial V \setminus \cup_{j=0}^{k} S_j, \qquad and \qquad \frac{\partial h}{\partial n} = \kappa_j \quad on \quad S_j, \tag{3.17}$$

where κ_j for $j = 0, ..., k$ are uniquely determined constants.

Proof. The proof of (3.16) for the case $\gamma = 1$ is standard. Indeed, let $\{w_i\}_{1=1}^{\infty}$ be a minimizing sequence satisfying $\lim_{i\to\infty} I_1[w_i] = J_1[\vec{F}]$. Then $\{w_i\}_{i=1}^{\infty}$ is bounded in $H^1(V)$. Therefore, there exists a subsequence $\{w_i\}$ and a function $v \in H_1(V)$ such that $w_i \rightharpoonup v \quad \text{in} \quad H^1(V)$.

Since $P_j(f)$ is a continuous functional on $H^1(V)$ in the strong topology, it is also continuous in the weak topology. In particular, $\mathcal{A}_{V,\vec{F}}$ is closed in the weak topology of $H_1(V)$ so $v \in \mathcal{A}_{V,\vec{F}}$. The lower semicontinuity of I_1 implies that $I_1[v] = J_1(\vec{F})$. Moreover, v is unique because I_1 is convex.

It remains to prove that v satisfies the boundary conditions in (3.17). Since v is a minimizer in $\mathcal{A}_{V,\vec{F}}$, it follows that

$$0 = \int_V (\nabla v \cdot \nabla w + vw)\, \mathrm{d}\mathbf{x} = \int_{\partial V \setminus \cup_{j=0}^{k} S_j} w \frac{\partial v}{\partial n}\, \mathrm{d}\xi + \int_{\cup_{j=0}^{k} S_j} w \frac{\partial v}{\partial n}\, \mathrm{d}\mathbf{s}.$$

for any $w \in \mathcal{A}_{V,\vec{0}}$. The first term in the last expression is thus zero only if $\partial v/\partial n = 0$ on $\partial V \setminus \cup_{j=0}^{k} S_j$ in the weak sense. Since the average of the test function w is zero on each sector ($P_j(w) = 0$), the second term is zero if $\partial v/\partial n = \kappa_j$ (in the weak sense) on S_j. Finally, the multipliers κ_j are uniquely determined due to the uniqueness of v for any $\vec{F}$.

The proof of (3.17) for the case $\gamma = 0$ is similar, except that we have to prove the bound in $L^2(V)$ of the minimizing sequence. Since V is a bounded Lipschitz domain, by [13, Theorem 5.5.1, and the remark in p. 286], the embedding $H^1(V) \to L^2(V)$ is compact. Hence, the spectrum of Helmholtz operator with the Neumann boundary condition for such domains is discrete. Its first eigenvalue is 1, and is a simple eigenvalue corresponding to the constant ground state. Hence, the Poincaré inequality

$$\int_V |v|^2 \mathrm{d}\mathbf{x} \le \Lambda_2^{-1} \int_V |\nabla v|^2 \mathrm{d}\mathbf{x}. \tag{3.18}$$

holds for all functions v perpendicular to the constant in $H_1(V)$, where Λ_2 is the second eigenvalue of the Neumann Laplacian on V.

We now repeat the argument for the case $\gamma = 1$, but restrict our domain to the domain of all functions in $v \in \mathcal{A}_{V,\vec{F}}$ which are perpendicular to the constant. The minimizer u obtained in this way satisfies $P_j(u) = F_j + \kappa$ for some $\kappa \in \mathbb{R}$ and $j \in \{0, \ldots, k\}$. Then $u - \kappa \in \mathcal{A}_{V,\vec{F}}$. $\square$

Let now $\delta > 0$, and set $\delta V := \{\delta x \;;\; x \in V\}$ the scaled inflated vertex, where we assume (as usual) that the center of V is in the origin. The sections of δV

are scaled accordingly, and we denote them by δS_j, $0 \le j \le k$. We define, correspondingly, the averaging operator on δS_j for $h \in H^1(\delta V)$:

$$P_j^\delta(h) := \frac{1}{\delta^{N-1}|S_j|}\int_{\delta S_j} h\,\mathrm{d}\mathbf{s}, \tag{3.19}$$

and

$$\mathcal{A}^\delta_{V,\vec F} := \left\{ g \in H^1(\delta V) \mid P_j^\delta(g) = F_j \quad \forall j = 0, ..., k \right\} . \tag{3.20}$$

Using Lemma 3.4, the following lemma is proved analogously to the proof of the second part of Lemma 3.3.

Lemma 3.5. *There exist $\beta^A > 0$ and $\beta^B > 0$ such that for all $f \in \mathcal{A}^\delta_{V,\vec F}$*

$$\int_{\delta V} |\nabla f|^2\,\mathrm{d}\mathbf{x} \ge \delta^{N-2}\beta^A|\vec F \llcorner \vec 1|^2, \tag{3.21}$$

and

$$\int_{\delta V} (|\nabla f|^2 + |f|^2)\,\mathrm{d}\mathbf{x} \ge \beta^B\left(\delta^{N-2}|\vec F \llcorner \vec 1|^2 + \delta^N|\vec F|^2\right). \tag{3.22}$$

4. Discreteness of the spectrum on T_1 and T_N

In this section we study the discreteness of the spectrum of width-weighted operators on T_1 and Schrödinger operators on T_N.

4.1. **Discreteness of the spectrum for weighted operators on T_1.**

In [4], Carslon has shown that the spectrum of the Laplacian on a connected metric graph G of finite volume which has a compact completion $\overline{G}$ is purely discrete. Solomyak [22] has extended Carlson's result to regular trees of a finite radius R.

Theorem 4.1 (Solomyak [22])**.** *Let T_1 be a radial tree such that $R(T_1) < \infty$ and its branching function is uniformly bounded. Let $W(x)$ be a radially symmetric measurable real valued function which is bounded below. Then the spectrum of $-\Delta + W$ on T_1 is purely discrete.*

Outline of Solomyak's proof. Solomyak constructed a family of weighted operators $\{A_{W,v}\}$ which are defined on the intervals $[t_v, R) \subseteq \mathbb{R}$, where t_v is the distance of a vertex v from the root O. The operators $A_{W,v}$ are defined as the selfadjoint operators in $L^2_g(t_v, R)$, associated with the quadratic form

$$a_{W,v}[u] := \int_{t_v}^R \left[|u'(t)|^2 + W(t)|u(t)|^2\right] g(t)\,\mathrm{d}t \qquad u \in C_0^\infty(t_v, R), \tag{4.1}$$

where g is the counting function. Using a decomposition of functions in $H(T_1)$ into symmetric functions on subtrees [14] (which implies the spectral decomposition of the Laplacian to these operators), Solomyak showed the equivalence between the discreteness of the spectrum of the Laplacian on T_1 and the discreteness of the spectrum of $A_{W,v}$ on $[t_v, R)$ for all vertices $v \in T_1$. Using a theorem of Birman and Borzov [3] and a certain change of variables, it is then shown that all the operators $A_{W,v}$ have discrete spectra. The proof of this part relies on the monotonicity of the counting function g. □

The basic ingredient in Solomyak's proof, namely the spectral decomposition into the subspaces of functions which are symmetric on subtrees, still holds if one adds weight functions which are symmetric in generations (see [14, 22] for details). The Schrödinger-type operators we consider in this section are defined

on the *weighted* tree T_1 and involve a pair of symmetric weight functions ρ_α, ρ_β and a symmetric potential W:

$$\overline{L}_{\alpha,\beta} u := -\rho_\beta^{-1} \frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_\alpha \frac{\mathrm{d}u}{\mathrm{d}t}\right) + Wu. \tag{4.2}$$

The spectral decomposition of $\overline{L}_{\alpha,\beta}$ is obtained by reducing these operators to the space of functions which are symmetric on all subtrees. The restriction of $\overline{L}_{\alpha,\beta}$ to the symmetric subtree $T_{1,v}$ with a root $v \in T_1$ are obtained by the quadratic form

$$a_{\alpha,\beta,W,v}[u] = \int_{t_v}^{R} \left[|u'(t)|^2 \rho_\alpha + W(t)|u(t)|^2 \rho_\beta\right] g(t)\,\mathrm{d}t. \tag{4.3}$$

and the associated operator in $L_2([t_v, R))$ is denoted by $A_{\alpha,\beta,W,v}$. To extend the result of Solomyak to the weighted tree we should show that $A_{\alpha,\beta,W,v}$ has a discrete spectrum for each vertex $v \in T_1$. Even though (4.3) seems very close to (4.1), the counting function g in (4.1) is replaced by $g\rho_\alpha$ and $g\rho_\beta$ in (4.3), and these functions are not necessarily monotone. We prove the discreteness of $\overline{L}_{\alpha,\beta}$ under the weaker condition that $g\rho_\alpha$ and $g\rho_\beta$ are uniformly bounded from below:

Theorem 4.2. *Let T_1 be a one-dimensional tree, whose radius R is finite. Assume that $0 < \rho < 1$ is a symmetric weight function on T_1, that $\rho_\alpha \asymp \rho$ and $\rho_\beta \asymp \rho$ are symmetric weight functions. Suppose that there exists a constant $C > 0$ so that*

$$Cg(s)\rho(s) < g(t)\rho(t) \qquad \textit{for all} \quad s \le t \le R(T_1). \tag{4.4}$$

Then the spectrum of the width-weighted operator $\overline{L}_{\alpha,\beta}$ on T_1 is purely discrete.

We use the following general Lemma of Lewis.

Lemma 4.3 ([11, Lemma 1]). *Let D be a domain in $\mathbb{R}^N$. Let h be a strictly positive symmetric closed form whose domain $H_{\mathrm{h}}(D)$ is dense in the Hilbert space $L^2_w(D)$ for a positive weight function w on D.*
Suppose that D is the union of an increasing sequence of open sets $\{D_j\}$, for which the identity injection $i_j : H_{\mathrm{h}}(D_j) \to L^2_w(D_j)$ is compact.
If there is a positive-valued function $p(x)$ on D and a sequence of positive numbers $\varepsilon_j \to 0$ as $j \to \infty$ such that

$$w(x)p(x)^{-1} < \varepsilon_j \qquad \textit{for almost every } x \in D\setminus D_j,$$

and

$$\int_{D\setminus D_j} p(x)|u(x)|^2 \mathrm{d}x \le \mathrm{h}[u,u] \qquad \textit{for all } u \in H_{\mathrm{h}}, \tag{4.5}$$

then the selfadjoint operator on $L^2_w(D)$ associated with the Friedrich extension of h has a purely discrete spectrum.

Proof of Theorem 4.2. We only need to show that for any $v \in T_1$ the operator $A_{\alpha,\beta,W,v}$ associated with (4.3) has a discrete spectrum. Evidently, it is enough to show it for $v = O$. For this, we use Lemma 4.3 with the quadratic form $h = a_{\alpha,\beta}$ on $L^2_{\rho_\beta} = L^2((0,R), \rho_\beta\,\mathrm{d}t)$. We set $D = [0,R)$, $D_j = [0, t_j)$. We denote

$$p(\theta) := \frac{\rho(\theta)g(\theta)}{R(R - |\theta|)}\ .$$

Since $0 < \rho < 1$ and $g\rho$ satisfies (4.4), it follows that p satisfies the assumptions of Lemma 4.3. By our assumptions $\rho_\alpha \asymp \rho_\beta \asymp \rho$, therefore it is sufficient to prove that for all $u \in C_0^1([0,R))$ and $0 < j < R$ we have

$$\int_j^R p(\theta)|u(\theta)|^2 \,\mathrm{d}\theta \le C \int_0^R |u'(\theta)|^2 \rho(\theta) g(\theta)\,\mathrm{d}\theta.$$

In fact, for any $j < \theta < R$

$$|u(\theta)|^2 = \left|\int_\theta^R u'(t)\mathrm{d}t\right|^2 \le |R-\theta| \int_\theta^R |u'(t)|^2 \mathrm{d}t.$$

Then

$$\begin{aligned}\int_j^R p(\theta)|u(\theta)|^2\,\mathrm{d}\theta &\le \int_j^R |R-\theta| p(\theta)\left[\int_\theta^R |u'(\zeta)|^2\,\mathrm{d}\zeta\right]\mathrm{d}\theta \\ &\le \frac{1}{R}\int_j^R \left[\rho(\theta)g(\theta)\int_\theta^R |u'|^2\,\mathrm{d}\zeta\right]\mathrm{d}\theta \le \frac{C}{R}\int_j^R\left[\int_\theta^R \rho g|u'|^2\,\mathrm{d}\zeta\right]\mathrm{d}\theta \\ &\le C\int_0^R \rho g |u'|^2\,\mathrm{d}\theta\ .\end{aligned}$$

□

4.2. **Discreteness of the spectrum for operators on T_N.** As we have mentioned, we are interested in spectral properties of Schrödinger operators on the N-dimensional tree T_N^ε. It is well known that the Laplacian on a compact manifold with a smooth boundary, and with standard (regular) boundary conditions has a pure point spectrum. However, since we wish to address also the problem of the discreteness of the spectrum for nonsmooth trees with an infinite volume, we cannot implement the classical theory. Instead, we use Lemma 4.3 to prove the discreteness of the spectrum of Schrödinger operator on T_N^ε with a finite radius.

Recall Definition 2.2 of the straightened tree $\hat{T}_N$. By Corollary 2.3 we can assign $\hat{T}_N$ a global coordinate system to the tree, namely $(\vec{s},\theta)$, where $\vec{s} \in \hat{\Omega}$ and $\theta \in [0,\hat{R})$. We pose the following assumption.

Assumptions 4.4. There exists a C^1-diffeomorphism $\mathcal{G} : T_N \to \hat{T}_N$. We denote by $\mathcal{F}$ its inverse, so that, $\mathcal{F} : \hat{T}_N \to T_N$. Denote by J the Jacobian of $\mathcal{F}$. We assume that there is a constant $C > 0$ such that

$$\left|\frac{\partial \mathcal{F}(\vec{s},\theta)}{\partial\theta}\right| \le C \qquad \forall (\vec{s},\theta) \in \hat{T}_N, \tag{4.6}$$

$$0 < J(\vec{s},\theta_1) \le CJ(\vec{s},\theta_2) \qquad \forall \theta_1 \le \theta_2. \tag{4.7}$$

We have in mind the following two-dimensional example.

Example 4.5. Let T_2 be a two-dimensional binary symmetric tree constructed by gluing rectangles and triangles (see Figure 4). Let the length of a rectangle in generation j be r^j and its width be d^j, where $r, d \in (0,1)$. Clearly, T_2 can be embedded in $\mathbb{R}^3$ to avoid overlapping of the edges. Notice that such a tree may have an infinite area (though its radius is finite). Indeed, the area of such a tree is given by $\sum_{j=1}^\infty [(2dr)^j + \beta d^{2j}]$ for some constant β, hence for any choice of $r < 1, d < 1$ such that $rd > 1/2$, the area is infinite. Let us denote by P_j the pentagon constructed by gluing a rectangle and triangle in the j generation, and

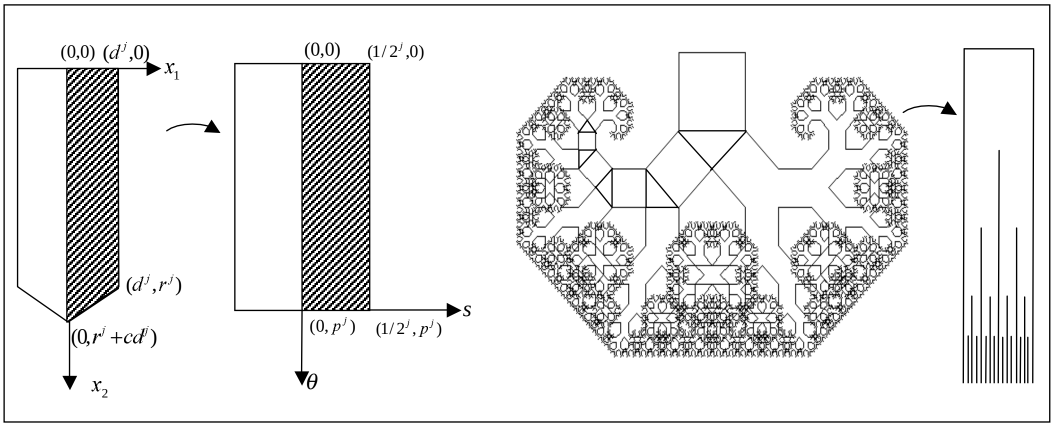

FIGURE 4. The transformation of T_2 to $\widehat{T}_2$.

by $P_{j,l}$ for $l = 1, 2$ its partition into two symmetric quadrangles. We assume that the coordinates of the vertices of the quadrangle $P_{j,l}$, $(x_{1,a}, x_{2,a})$ for $a = 1, ..., 4$, are given (up to translations) by $(0, 0), (d^j, 0), (d^j, r^j), (0, r^j + cd^j)$ respectively for a constant c.

Let $p = (R - 1)/R$, where R is the radius of the original tree (p is chosen such that $\sum_{j=0}^{\infty} p^j = R$). In particular, $r < p < 1$. A transformation of a rectangle whose vertices are at $(0, 0), (1/2^j, 0), (1/2^j, p^j), (0, p^j)$ onto $P_{j,l}$ can be written in the form

$$x_1(\theta, s) = (2d)^j s, \quad \text{and} \quad x_2(\theta, s) = \frac{r^j}{p^j}\theta + c\frac{d^j}{p^j}\theta - c\frac{2^j d^j}{p^j} s\theta.$$

An elementary calculation shows that if $d \leq p$, then $|\partial x_2/\partial\theta| \leq 1 + c$. Note that $\partial x_1/\partial s = (2d)^j$ is not bounded for $d > 1/2$, which means that the total width of the tree is unbounded. However, the condition $d > 1/2$ ensures the possibility of gluing together the connectors and the edges of this tree.

Assumptions 4.6. Let $\hat{V} \subset \hat{\Omega} \times [0, 1]$ be an inflated vertex of the straightened tree, where $\hat{S}_0 := \hat{\Omega} \times \{0\}$, $\hat{S}_j \cong k^{-1/N}\hat{\Omega} \times \{1\}$, $1 \leq j \leq k$ the corresponding sections. Let V be the inflated vertex of a given tree T_N, and $S_j \subset \partial V$, $0 \leq j \leq k$ the corresponding sections. We assume that there exists a C^1-diffeomorphism $\mathcal{F} = \mathcal{F}(\vec{s}, \theta) : \hat{V} \to V$ so that $\mathcal{F}(\hat{S}_j) = S_j$ for $0 \leq j \leq k$, and such that

$$\left|\frac{\partial \mathcal{F}}{\partial \theta}\right| < C, \qquad \text{and} \qquad 0 < J(\vec{s}, \theta_1) \leq C J(\vec{s}, \theta_2) \qquad \text{if } \theta_1 \leq \theta_2$$

hold on V for some $C > 0$, where J is the Jacobian of $\mathcal{F}$.

Remark 4.7. *Assumptions 4.6 imply Assumptions 4.4.*

Theorem 4.8. *Under Assumptions 4.4 (resp. Assumptions 4.6), the Laplace operator on T_N as defined in Section 2.6, has a purely discrete spectrum.*

Proof. Let $\mathcal{G} : T_N \to \hat{T}_N$ be the inverse C^1-mapping of $\mathcal{F}$ which is defined in Assumptions 4.4, and set $\mathcal{G}(\mathbf{x}) := (\theta(\mathbf{x}), \vec{s}(\mathbf{x})) \in \hat{T}_N$. Denote by J the Jacobian of $\mathcal{F}$. Let $\hat{T}_{N,j} \subset T_N$ be the finite subtree

$$\hat{T}_{N,j} := \left\{(\theta, \vec{s}) \in \hat{T}_N \mid \theta < \theta_j\right\},$$

where $\theta_j \nearrow \hat{R}$, and $\hat{R}$ is the radius of $\hat{T}_N$. Let

$$T_{N,j} := \mathcal{F}\left(\hat{T}_{N,j}\right), \tag{4.8}$$

and set

$$p(\mathbf{x}) := \frac{1}{C^2\hat{R}|\hat{R} - \theta(\mathbf{x})|} .$$

We wish to use Lemma 4.3 with $D_j \equiv T_{N,j}$. This Lemma requires the compactness of the identity injection $i_k : H^1(D_j) \to L^2(D_j)$. Although the boundary of $D_j = T_{N,j}$ is not C^1, this injection is still compact. Indeed, the embedding $i : H^1(D) \to L^2(D)$ is compact for a bounded domain D which has the (inner) cone property (see [13, Theorem 5.5.1], and the remark on p. 286 therein).

By Lemma 4.3, it is sufficient to prove for the Laplacian that

$$\int_{T\setminus T_{N,j}} p(\mathbf{x})|u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \le \int_{T_N} |\nabla u|^2 \mathrm{d}\mathbf{x} \tag{4.9}$$

for all $u \in C^1(T_N)$ that vanish on the 'top' of T_N and outside $T_{N,j}$ for some $j \ge 1$. Let u be such a test function, and let $v(\theta, \mathbf{s}) = u(\mathbf{x})$. Then

$$|u(\mathbf{x})|^2 = |v(\theta,\mathbf{s})|^2 = \left|\int_\theta^{\hat{R}} \frac{\partial v}{\partial \vartheta}\mathrm{d}\vartheta\right|^2 \le |\hat{R} - \theta| \int_\theta^{\hat{R}} \left|\frac{\partial v}{\partial \vartheta}\right|^2 \mathrm{d}\vartheta. \tag{4.10}$$

Using the definition of the function p, (4.6), (4.7), (4.10), and Fubini's theorem, we obtain,

$$\begin{aligned} \int_{T_N\setminus T_{N,j}} p(\mathbf{x})|u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} &= \int_{\widehat{T}_N\setminus \widehat{T}_{N,j}} p(\theta,\mathbf{s})|v(\theta,\mathbf{s})|^2 J \mathrm{d}\xi \\ &\le \int_{\widehat{T}_N\setminus \widehat{T}_{N,j}} \frac{1}{C^2\hat{R}} \left(\int_\theta^{\hat{R}} \left|\frac{\partial v}{\partial \vartheta}\right|^2 \mathrm{d}\vartheta\right) J\mathrm{d}\xi \le \int_{\theta_j}^{\hat{R}}\int_{\hat{\Omega}} \frac{1}{C^2\hat{R}} \left(\int_\theta^{\hat{R}} \left|\frac{\partial v}{\partial \vartheta}\right|^2 \mathrm{d}\vartheta\right) J(\mathbf{s},\theta)\mathrm{d}\mathbf{s}\mathrm{d}\theta \\ &\le \int_{\theta_j}^{\hat{R}}\int_{\hat{\Omega}} \frac{1}{C\hat{R}} \left(\int_\theta^{\hat{R}} \left|\frac{\partial v}{\partial \vartheta}\right|^2 J(\mathbf{s},\vartheta)\,\mathrm{d}\vartheta\right)\mathrm{d}\mathbf{s}\mathrm{d}\theta \le \frac{1}{C\hat{R}}\int_{\theta_j}^{\hat{R}} \left(\int_{\hat{\Omega}}\int_0^{\hat{R}} \left|\frac{\partial v}{\partial \vartheta}\right|^2 J(\mathbf{s},\vartheta)\mathrm{d}\vartheta\mathrm{d}\mathbf{s}\right)\mathrm{d}\theta \\ &\le \frac{1}{C}\int_{\widehat{T}_N} \left|\frac{\partial v}{\partial \vartheta}\right|^2 J\mathrm{d}\xi = \frac{1}{C}\int_{T_N} \left|\sum_{i=1}^n \frac{\partial u}{\partial x_i}\frac{\partial x_i}{\partial \theta}\right|^2 \mathrm{d}\mathbf{x} \le \int_{T_N} |\nabla u|^2 \mathrm{d}\mathbf{x}. \end{aligned} \tag{4.11}$$

Since (4.9) is satisfied, the spectrum of the Laplacian on T_N is purely discrete. □

Remark 4.9. A similar proof applies for a Schrödinger operator on T_N with a bounded from below potential.

4.3. **Further results.** In this subsection we present two lemmas asserting that the L^2-norm of functions which are bounded in $H^1_{0,\rho_\alpha}(T_1)$ and in $H^1_0(T^\varepsilon_N)$ is concentrated on compact sets. These lemmas will be used in Section 6. Their proofs are similar to those of theorems 4.2 and 4.8, and therefore they are omitted.

Lemma 4.10. *Assume that T_1 satisfies the assumptions of Theorem 4.2. Suppose that there exists a weight function $0 < \rho < 1$, which is constant on each edge of T_1, such that $\rho \asymp \rho_\alpha$ and $\rho \asymp \rho_\beta$ with a constant c. Denote $T_{1,j} = \{\mathrm{gen}(e) \le j\}$.*

(1) *Let $R(j)$ be the radius of the maximal (connected) subtree in $T\backslash T_{1,j}$. Then*

$$\int_{T_1\backslash T_{1,j}} |u|^2\rho_\beta \,\mathrm{d}\theta \leq \frac{c^2}{C}R(j)^2\int_{T_1}|u'|^2\rho_\alpha\,\mathrm{d}\theta \qquad \forall u\in H^1_{0,\rho_\alpha}(T_1).$$

(2) *Let $\overline{\mathcal{V}}^\varepsilon := \bigcup_{v\in T_1}\overline{V}^\varepsilon(v)$. Then*

$$\int_{\overline{\mathcal{V}}^\varepsilon} |u|^2\rho_\beta\,\mathrm{d}\theta \leq O(\varepsilon)\int_{T_1}|u'|^2\rho_\alpha\,\mathrm{d}\theta \qquad \forall u\in H^1_{0,\rho_\alpha}(T_1).$$

Lemma 4.11. *Assume that T_N satisfies the assumptions of Theorem 4.8.*

(1) *Let $T_{N,j}$ as defined in (4.8), and let $R(j) := \hat{R}-\theta_j$. Then*

$$\int_{T_N\backslash T_{N,j}} |u(\mathbf{x})|^2\,\mathrm{d}\mathbf{x} \leq C^2R(j)^2\int_{T_N}|\nabla u|^2\,\mathrm{d}\mathbf{x} \qquad \forall u\in H^1_0(T_N). \tag{4.12}$$

(2) *Let $\mathcal{V}^\varepsilon := \bigcup_{v\in T_1}V^\varepsilon(v)$. Then*

$$\int_{\mathcal{V}^\varepsilon} |u(\mathbf{x})|^2\,\mathrm{d}\mathbf{x} \leq O(\varepsilon)\int_{T^\varepsilon_N}|\nabla u|^2\,\mathrm{d}\mathbf{x} \qquad \forall u\in H^1_0(T^\varepsilon_N). \tag{4.13}$$

5. Convergence of the spectra of width-weighted operators

In this section we estimate the eigenvalues of the width-weighted operators on T_1 (defined in Section 2.6), for the case where the weight functions and the potential depend on ε, and pointwise converge as ε tends to 0. We treat the weight functions and potential term as convergent sequences of functions of ε. Hence, throughout this section we set $\varepsilon := 1/n$, where $n\in\mathbb{N}$, and denote the weights and potentials by $\rho_{\alpha,n}$, $\rho_{\beta,n}$ and W_n. Accordingly, the corresponding operators are denoted by $A_{\alpha,\beta,n}$, or A_n for short. We assume that $\rho_{\alpha,n}$ and $\rho_{\beta,n}$ converge to a mutual weight function, which we denote by ρ. We denote by W the limit potential of the sequence W_n. We also treat the spaces $\{H^1_{0,n}(T_1)\} := \{H^1_{0,\rho_{\beta,n}}(T_1)\}$ as a spaces sequence, with a "limiting space" $H^1_{0,\rho}(T_1)$. Let $\{L^2_n(T_1)\}_{n=1}^\infty$ and $L^2_\rho(T_1)$ be the corresponding L^2 spaces. Using these notations, we study the asymptotic behavior of the eigenvalues of A_n as $n\to\infty$.
Throughout this section we assume that the following conditions are satisfied:

Assumptions 5.1. (1) T_1 has a finite radius.

(2) **Assumptions on the weight functions:** $\{\rho_{1,n}\}_{n=1}^\infty$ and $\{\rho_{2,n}\}_{n=1}^\infty$ are positive bounded weight functions sequences in $L^1_{\mathrm{loc}}(T_1)$, such that $\rho_{1,n}\asymp\rho$ and $\rho_{2,n}\asymp\rho$ with the same constant c (so the spaces $H^1_{0,n}(T_1)$ and $H^1_{0,\rho}(T_1)$ are equivalent for all $n\in\mathbb{N}$). Moreover, for any neighborhood U containing all the vertices of T_1 and a given compact set $K\Subset T_1$, we have $\rho_{1,n}=\rho_{2,n}=\rho$ in $(T_1\cap K)\setminus U$ for all sufficiently large n.

(3) **Assumptions on the potential terms:** $\{W_{T_1,n}\}_{n=1}^\infty$ is a sequence of real valued radially symmetric potentials on T_1, for which there exists a positive constant C_W such that $|W_{T_1,n}|_{L^\infty_\rho(T_1)}\leq C_W$. Moreover, $\{W_{T_1,n}\}_{n=1}^\infty$ converges almost surely (and hence in $L^1_{\rho,\mathrm{loc}}(T_1)$) to a potential W, which satisfies $|W|_{L^\infty_\rho(T_1)}\leq C_W$. Without loss of generality, we assume that $W_{T_1,n}>1$ for all $n\in\mathbb{N}$.

Under Assumptions 5.1, we show that the eigenvalues of the operators A_n converge, as $n\to\infty$, to the eigenvalues of the limit operator A. Here the operators

A_n are defined by the quadratic forms on $H_0^1(T_1) \times H_0^1(T_1)$:

$$< A_n u, \phi >_n := \int_{T_1} (u'\bar{\phi}'\rho_{1,n} + W_{T_1,n} u\bar{\phi}\,\rho_{2,n})\,\mathrm{d}t, \tag{5.1}$$

while the limit operator A is defined, similarly, by

$$< Au, \phi > := \int_{T_1} (u'\bar{\phi}' + Wu\bar{\phi})\rho\,\mathrm{d}t. \tag{5.2}$$

This result is stated in Corollary 5.4. Notice that since ρ is constant on each edge, the difference between the derivatives part of A and the Laplacian is manifested by the Kirchhoff condition.

In order to prove the convergence of the spectrum, we need the following lemmas, whose proofs are given later.

Lemma 5.2. *For $n \in \mathbb{N}$, consider operators A_n of the form* (5.1) *which satisfy Assumptions 5.1. Let $\{u_n\} \subset H^1_{0,\rho}(T_1)$ be a sequence of normalized eigenfunctions of A_n which converges weakly in $H^1_{0,\rho}(T_1)$ to u. Let λ_n be the sequence of corresponding eigenvalues of A_n. If $\lim_{n\to\infty}\lambda_n = \lambda$, then $Au = \lambda u$, and $u \neq 0$ is an eigenfunction of the operator A defined in (5.2) with eigenvalue λ. Moreover, $\{u_n\}$ also converges locally uniformly to u.*

Lemma 5.3. *Consider operators A, and A_n for $n \in \mathbb{N}$, of the form* (5.1) *and* (5.2) *respectively which satisfy Assumptions 5.1. Assume also that $A_n^{-1} : L^2_\rho(T_1) \to H^1_{0,\rho}(T_1)$ have uniform bounded norms. Suppose that $u \neq 0$, and $Au = \lambda u$ in $L^2_\rho(T_1)$. For each $n \in \mathbb{N}$, let w_n be the solution of the equation $A_n w_n = \lambda u$ in $L^2_\rho(T_1)$. Then $\{w_n\}$ has subsequence that we continue denoting by $\{w_n\}$, which converges to u weakly in $H^1_{0,\rho}(T_1)$ and strongly in $L^2_\rho(T_1)$. Moreover, $\{w_n\}$ also converges locally uniformly to u.*

Theorem 5.4. *Let $\{\rho_{1,n}\}_{n=1}^\infty, \{\rho_{2,n}\}_{n=1}^\infty$ and $\{W_{T_1,n}\}_{n=1}^\infty$ be sequences of weight functions and potentials on T_1 satisfying Assumptions 5.1. Assume, in addition, that $\{W_{T_1,n}\}_{n=1}^\infty$ are continuous functions and that $\{\rho_{1,n}\}_{n=1}^\infty$ and $\{\rho_{2,n}\}_{n=1}^\infty$ equal ρ except, at most, for $O(|e_j|/n)$ neighborhoods of vertices in generation j. Let a sequence of operators A_n and a limit operator A be defined by* (5.1) *and* (5.2) *respectively. We denote by $\lambda_{m,n}$ the m-th eigenvalue of A_n, and by λ_m the m-th eigenvalue of A. Then*

$$\lim_{n\to\infty} \lambda_{m,n} = \lambda_m.$$

Proof. We adapt Attouch's proof of [2, Theorem 3.71]. Since $\rho_{1,n} \asymp \rho$ and $\rho_{2,n} \asymp \rho$ with a positive constant c, and $|W_{T_1,n}|$ and $|W|$ are bounded by C_W, we have for all $u \neq 0$ that the Rayleigh quotients satisfy

$$R_n(u) := \frac{<A_n u, u>_n}{<u,u>_n} = \frac{\int_{T_1}(|u'|^2\rho_{1,n} + W_{T_1,n}|u|^2\rho_{2,n})\,\mathrm{d}\theta}{\int_{T_1}|u|^2\rho_{2,n}\,\mathrm{d}\theta} \leq c^2\frac{<Au,u>}{<u,u>_\rho} + 2c^2 C_W, \tag{5.3}$$

and similarly

$$\frac{< A_n u, u >_n}{< u, u >_n} \geq \frac{1}{c^2}\frac{< Au, u >}{< u, u >_\rho} - 2\frac{1}{c^2}C_W.$$

Fix $l \in \mathbb{N}$, by the min-max principle we obtain

$$\frac{1}{c^2}(\lambda_l - 2C_W) \leq \lambda_{l,n} \leq c^2(\lambda_l + 2C_W), \tag{5.4}$$

so, $\{\lambda_{l,n}\}$ is a bounded sequence. Therefore, there exists a subsequence of $\{\lambda_{l,n}\}$ (that we keep denoting by $\{\lambda_{l,n}\}$), and $\widehat{\lambda}_l \in \mathbb{R}$ such that $\lambda_{l,n} \to \widehat{\lambda}_l$.

We claim that there exists an eigenfunction $\widehat{u}_l$ such that $A\widehat{u}_l = \widehat{\lambda}_l \widehat{u}_l$, i.e., $\{\widehat{\lambda}_l\} \subseteq \{\lambda_j\}$. Indeed, let $\{u_{l,n}\}$ be the orthonormal sequence of eigenfunctions of A_n that correspond to $\{\lambda_{l,n}\}$. We assume that $\|u_{l,n}\|_n = 1$. Then

$$\int_{T_1} |\,(u_{l,n})'\,|^2 \rho_{1,n}\,\mathrm{d}\theta = \int_{T_1} (\lambda_{l,n} - W_{T_1,n})|u_{l,n}|^2 \rho_{2,n}\,\mathrm{d}\theta \le \lambda_{l,n} + C_W.$$

It follows that $\{u_{l,n}\}$ is bounded in $H^1_{0,\rho}$. The weak sequential compactness implies that $\{u_{l,n}\}$ has a subsequence $\{u_{l,n}\}$ which converges weakly in $H^1_{0,\rho}(T_1)$. We denote its limit by $\widehat{u}_l$. By Lemma 5.2, $\widehat{u}_l \neq 0$, $A\widehat{u}_l = \widehat{\lambda}_l \widehat{u}_l$ and the convergence is locally uniform. In particular, $\{\widehat{\lambda}_l\} \subseteq \{\lambda_j\}$. Moreover, (5.4) implies that $\{\widehat{\lambda}_l\}$ is an infinite sequence, and since $\{\widehat{\lambda}_l\} \subseteq \{\lambda_j\}$, we have $\lim_{l\to\infty} \widehat{\lambda}_l = \infty$.

Let us now show that $\{\lambda_j\} \subseteq \{\widehat{\lambda}_l\}$. Assume that there exists an eigenvalue λ of A such that $\lambda \neq \widehat{\lambda}_l$ for all $l \in \mathbb{N}$, and let u be a corresponding eigenfunction of A such that $||u||_{L^2_\rho} = 1$.

Take $m \in \mathbb{N}$ such that $\lambda < \widehat{\lambda}_{m+1}$ for all limit values $\widehat{\lambda}_{m+1}$ of the sequence $\{\lambda_{m+1,n}\}$. Set $U_{m,n} = \mathrm{span}\{u_{1,n}, ..., u_{m,n}\}$. By the min-max principle,

$$\lambda_{m+1,n} = \min_{v\in U^{\perp}_{m,n}} R_n(v),$$

where R_n is defined in (5.3). Therefore, if we could find $v_n \in U^{\perp}_{m,n}$ satisfying

$$\lim_{n\to\infty} R_n(v_n) \le \lambda,$$

then we would arrive to a contradiction of the assumption $\lambda < \widehat{\lambda}_{m+1}$.

Let w_n be the solutions of the problem $A_n w_n = \lambda u$. The assumption $W_{T_1,n} > 1$ implies that A_n^{-1} are uniformly bounded, so $\{w_n\}$ is a bounded sequence in $L^2_\rho(T_1)$. By Lemma 5.3, up to a subsequence, $\{w_n\}$ converges to u, weakly in $H^1_{0,\rho}(T_1)$, strongly in $L^2_\rho(T_1)$, and also locally uniformly.

Let us show that $\lim_{n\to\infty} R_n(w_n) = \lambda$:

(5.5)
$$< A_n w_n, w_n >_n = \lambda < u, w_n >_n = \lambda \int_{T_1} [u\overline{w}_n(\rho_{2,n} - \rho) + u(\overline{w}_n - \overline{u})\rho]\,\mathrm{d}t + \lambda.$$

Since

$$\left|\int_{T_1} u\overline{w}_n(\rho_{2,n} - \rho)\,\mathrm{d}t\right|^2 = \left|\int_{T_1} u\overline{w}_n \rho \frac{(\rho_{2,n} - \rho)}{\rho}\,\mathrm{d}t\right|^2 \le \left\| u\left(\frac{\rho_{2,n} - \rho}{\rho}\right)\right\|^2_{L^2_\rho} \|w_n\|^2_{L^2_\rho}\ ,$$

Lebesgue's dominated convergence theorem implies that the first term of the right-hand side of (5.5) converges to zero, while the second term tends to zero due to the $L^2_\rho(T_1)$ convergence of $\{w_n\}$ to u. Therefore,

$$\lim_{n\to\infty} < A_n w_n, w_n >_n = \lambda. \tag{5.6}$$

Moreover,

$$< w_n, w_n >_n = \int_{T_1} \left[(|w_n|^2 - |u|^2)\rho_{2,n} + |u|^2(\rho_{2,n} - \rho) + |u|^2\rho\right]\,\mathrm{d}t. \tag{5.7}$$

The first terms in (5.7) converges to zero due to the strong convergence of w_n to u in $L^2_\rho(T_1)$. Indeed,

$$\left|\int_{T_1}\!\left(|w_n|^2-|u|^2\right)\rho_{2,n}\,\mathrm{d}t\right| \le \|w_n-u\|^{1/2}_{L^2_{\rho n}}\|w_n+u\|^{1/2}_{L^2_{\rho n}} \le C\,\|w_n-u\|^{1/2}_{L^2_\rho}\|w_n+u\|^{1/2}_{L^2_\rho}\,.$$

The second term in (5.7) converges to zero by Lebesgue's dominated convergence theorem. Hence, (5.6) and (5.7) imply that

$$\lim_{n\to\infty} R_n(w_n)=\lambda. \tag{5.8}$$

Define

$$v_n := w_n - \sum_{k=1}^{m} < w_n, u_{k,n} >_n u_{k,n}.$$

Fix $1\le k\le m$, and let $\widehat{u}_k$ be a weak limit of $u_{k,n}$. It follows (as above) that

$$\lim_{n\to\infty} < w_n, u_{k,n} >_n = \lim_{n\to\infty} < w_n, u_{k,n} >_\rho = < u, \widehat{u}_k >_\rho\,. \tag{5.9}$$

By the first part of the proof, $\widehat{u}_k$ is an eigenfunction of A, and by our assumption, its eigenvalue is not equal to λ. Therefore, $< u, \widehat{u}_k >_n = 0$ and by (5.9),

$$\lim_{n\to\infty} < w_n, u_{k,n} >_n = 0. \tag{5.10}$$

That implies that $\{v_n\}$ and $\{w_n\}$ share the same L^2-limit u.

Using (5.6) and (5.10), a direct calculation yields that

$$\lim_{n\to\infty} < A_n v_n, v_n >_n = \lambda, \quad \text{and} \quad \lim_{n\to\infty} < v_n, v_n >_n = 1.$$

Hence

$$\lim_{n\to\infty} R_n(v_n) = \lim_{n\to\infty} R_n(w_n) = \lambda.$$

By the definition of v_n, we have $< v_n, u_{k,n} >_{L^n_n} = 0$ for all $k=1,...,m$. Hence, the min-max principle implies that $R_n(v_n)\ge \lambda_{m+1,n}$. Therefore, $\lambda \ge \widehat{\lambda}_{m+1}$ for some limit value $\widehat{\lambda}_{m+1}$, which contradicts the assumption $\lambda < \min\{\widehat{\lambda}_{m+1}\}$. □

Remark 5.5. Let $\{u_n\}_{n=1}^\infty \subseteq H^1_{0,\rho}(T_1)\cap C^1(T_1)$ be a sequence which converges weakly to u in $H^1_{0,\rho}(T_1)$. It follows that $\{u_n\}$ is locally a bounded and equicontinuous sequence in $C(T_1)$. By Arzelà-Ascoli's theorem, $\{u_n\}_{n=1}^\infty$ has a subsequence that converges locally uniformly to a continuous function u.

Proof of Lemma 5.2. By Remark 5.5, $\{u_n\}$ has a subsequence which we continue denoting by $\{u_n\}$, that converges locally uniformly to u which is continuous on T_1. We claim: (1) $u\in \mathrm{Dom}(A)$, (2) $Au=\lambda u$ and , and (3) $u\neq 0$. The first two claims follow provided we prove

$$\int_{T_1}(u'\overline{\phi'}+Wu\overline{\phi})\rho\,\mathrm{d}\theta = \lambda\int_{T_1} u\overline{\phi}\rho\,\mathrm{d}\theta \qquad \forall \phi\in C^1_0(T_1). \tag{5.11}$$

Since $\{u_n\}$ are eigenfunctions of A_n, for each test function $\phi\in C^1_0(T_1)$,

$$\int_{T_1}(u_n'\overline{\phi'}\rho_{1,n}+W_{T_1,n}u_n\overline{\phi}\rho_{2,n})\,\mathrm{d}\theta = \int_{T_1}\lambda_n u_n\overline{\phi}\rho_{2,n}\,\mathrm{d}\theta. \tag{5.12}$$

By Lebesgue's theorem applied to $\rho_{1,n}$ and the $H^1_{0,\rho}$ bound of u_n,

$$\lim_{n\to\infty}\left|\int_{T_1}u_n'\overline{\phi'}(\rho-\rho_{1,n})\,\mathrm{d}\theta\right|^2 \le \lim_{n\to\infty}\int_{T_1}|\phi'|^2\frac{(\rho-\rho_{1,n})^2}{\rho}\,\mathrm{d}\theta\int_{T_1}|u_n'|^2\rho\,\mathrm{d}\theta = 0. \tag{5.13}$$

The weak convergence of $\{u_n\}$ to u in $H^1_{0,\rho}(T_1)$ implies that

$$\lim_{n\to\infty}\int_{T_1} u_n'\overline{\phi'}\rho\,\mathrm{d}\theta = \int_{T_1} u'\overline{\phi'}\rho\,\mathrm{d}\theta. \tag{5.14}$$

By similar arguments, the local uniform convergence of $\{u_n\}$ to u, and the a.s. convergence of $W_{T_1,n}$ and $\rho_{2,n}$ imply that

(5.15)

$$\lim_{n\to\infty}\lambda_n\int_{T_1} u_n\overline{\phi}\rho_{2,n}\,\mathrm{d}\theta = \lambda\int_{T_1} u\overline{\phi}\rho\,\mathrm{d}\theta, \text{ and } \lim_{n\to\infty}\int_{T_1} W_{T_1,n}u_n\overline{\phi}\rho_{2,n}\,\mathrm{d}\theta = \int_{T_1} Wu\overline{\phi}\rho\,\mathrm{d}\theta.$$

Now, (5.12)–(5.15) imply (5.11).

In order to show that $u\neq 0$, let $T_{1,k}=\{e\in T_1\,|\,\mathrm{gen}(e)\leq k\}$, and let $R(k)$ be the maximal radius of subtrees in $T_1\backslash T_{1,k}$. Recall that $\{u_n\}$ are eigenfunctions satisfying $\|u_n\|_{H^1_{0,\rho}(T_1)}=1$, the corresponding eigenvalues sequence $\{\lambda_n\}$ converges, the potential terms $\{W_n\}$ are bounded by a constant C_W for all $n\in\mathbb{N}$, and $\rho_{1,n}\asymp\rho\asymp\rho_{2,n}$. Therefore, using (5.13) and the arguments that eigenfunctions has $L^2_\rho(T_1)$ and $H^1_\rho(T_1)$ norms of the same order, we infer that there exist $\gamma,\delta>0$ so that, for n large enough, $||u_n||_{L^2_\rho(T_1)}\geq\gamma>0$ and $||u_n'||_{L^2_\rho(T_1)}\leq\delta$.

Therefore, by Lemma 4.10 (1) we have that

$$\int_{T_{1,k}}|u_n|^2\rho\,\mathrm{d}\theta = \int_{T_1}|u_n|^2\rho\,\mathrm{d}\theta - \int_{T_1\backslash T_{1,k}}|u_n|^2\rho\,\mathrm{d}\theta$$

$$\geq \int_{T_1}|u_n|^2\rho\,\mathrm{d}\theta - \frac{c^2}{C}R(k)^2\int_{T_1}|u_n'|^2\rho\,\mathrm{d}\theta \geq \gamma-\delta\frac{c^2}{C}R(k)^2.$$

Now, choose k large enough such that $\gamma-\delta[cR(k)]^2/C>0$. By the local uniform convergence of u_n to u, we obtain

$$0<\gamma-\delta\frac{c^2}{C}R(k)\leq\lim_{n\to\infty}\int_{T_{1,k}}|u_n|^2\rho\mathrm{d}\theta = \int_{T_{1,k}}|u|^2\rho\mathrm{d}\theta\leq\int_{T_1}|u|^2\rho\mathrm{d}\theta.$$

Therefore $u\neq 0$, and u is an eigenfunction of A. □

Proof of Lemma 5.3. Since $Au=\lambda u$ and A is invertible, it is sufficient to prove that $Aw=Av$ (and in particular that w is in the domain of A). But this is equivalent to

$$\langle Aw,\phi\rangle = \langle Au,\phi\rangle \tag{5.16}$$

for any function ϕ in a dense subset of $H^1_{0,\rho}(T_1)$. Recall that $w\in H^1_{0,\rho}(T_1)$ and $\langle Aw,\phi\rangle$ is defined by (5.2). Let us split the quadratic form (5.2) into

$$\langle Aw,\phi\rangle = \langle Aw,\phi\rangle^{(1)}+\langle Aw,\phi\rangle^{(2)},$$

where

$$\langle Aw,\phi\rangle^{(1)} := \int_{T_1} w'\bar{\phi}'\rho\,\mathrm{d}t, \quad \langle Aw,\phi\rangle^{(2)} := \int_{T_1} Ww\bar{\phi}\rho\,\mathrm{d}t. \tag{5.17}$$

Similarly, (5.1) is written as

$$\langle A_nw,\phi\rangle_n = \langle A_nw,\phi\rangle_n^{(1)}+\langle A_nw,\phi\rangle_n^{(2)},$$

where

$$\langle A_nw,\phi\rangle_n^{(1)} := \int_{T_1} w'\bar{\phi}'\rho_{1,n}\,\mathrm{d}t, \quad \langle Aw,\phi\rangle_n^{(2)} := \int_{T_1} Ww\bar{\phi}\rho_{2,n}\,\mathrm{d}t. \tag{5.18}$$

Let $\Phi(T_1)$ be the set of all functions $\phi \in C_0^2(T_1)$ which are constant in some neighborhood of any vertex $v \in T_1$.

We further observe

(1) For any $\phi \in \Phi(T_1)$ and sufficiently large n, $\phi' = 0$ whenever $\rho_{1,n} \neq \rho$ or $\rho_{2,n} \neq \rho$ by Assumption 5.1 (2). Hence, for a given $\phi \in \Phi(T_1)$

$$\langle A_n w_n, \phi \rangle_n^{(1)} = \langle A w_n, \phi \rangle^{(1)}.$$

for all sufficiently large n.

(2) Since w is the weak limit of w_n in $H^1_{0,\rho}(T_1)$ it follows by (1) that

$$\lim_{n\to\infty} \langle A_n w_n, \phi \rangle_n^{(1)} = \lim_{n\to\infty} \langle A w_n, \phi \rangle^{(1)} = \langle A w, \phi \rangle^{(1)}.$$

(3) By Assumption 5.1 and the strong convergence of w_n to w in $L^2_\rho(T_1)$ we obtain

$$\lim_{n\to\infty} \langle A_n w_n, \phi \rangle_n^{(2)} = \langle A w, \phi \rangle^{(2)} .$$

(4) By (2) and (3) we obtain

$$\lim_{n\to\infty} \langle A_n w_n, \phi \rangle_n = \langle A w, \phi \rangle .$$

for any $\phi \in \Phi(T_1)$.

(5) Since $A_n w_n = \lambda u = A u$ by assumption we obtain $\langle A_n w_n, \phi \rangle_n = \lambda \langle u, \phi \rangle_n = \langle A u, \phi \rangle_n$. Since $\rho_{n,1} \to \rho$ in measure, it follows that

$$\langle A w, \phi \rangle = \lim_{n\to\infty} \langle A_n w_n, \phi \rangle_n = \lambda \langle u, \phi \rangle_\rho = \langle A u, \phi \rangle .$$

So, (5.16) is proved for any $\phi \in \Phi(T_1)$. The proof is completed by observing that $\Phi(T_1)$ is clearly dense in $H^1_{0,\rho}(T_1)$. □

6. ε-dependent bounds for the eigenvalues of N-dimensional tree

In this section, we consider the spectrum of the Schrödinger operator

$$L_\varepsilon := -\Delta + W_{T_N^\varepsilon}$$

on T_N^ε, where N is dimension of the tree, and $W_{T_N^\varepsilon}$ is a continuous bounded potential on T_N^ε. Without loss of generality, we assume that $W_{T_N^\varepsilon} \geq 0$.

We prove that the eigenvalues of L_ε are bounded from above and below by functions $\phi_Q^\varepsilon, \phi_P^\varepsilon$ of the eigenvalues of weighted operators A_ε on T_1 of the form

$$A_\varepsilon := -\frac{1}{\rho_{b,\varepsilon}} \frac{\mathrm{d}}{\mathrm{d}\theta} \left(\rho_{a,\varepsilon} \frac{\mathrm{d}}{\mathrm{d}\theta} \right) + W_{T_1,\varepsilon} \,, \tag{6.1}$$

for a suitable choice of weight functions $\rho_{a,\varepsilon}, \rho_{b,\varepsilon}$ and a potential $W_{T_1,\varepsilon}$ on T_1 of the form

$$W_{T_1,\varepsilon}(\theta) := \begin{cases} \dfrac{\int_{\Omega_e^\varepsilon} W_{T_N^\varepsilon}(\theta, \mathbf{s})\, \mathrm{d}\mathbf{s}}{|\Omega_e^\varepsilon|} & \theta \in \overline{E}^\varepsilon(e), \\ \sum_{e\in N(v)} b_e \psi_{(e)}(\theta) & \theta \in \overline{V}^\varepsilon(v), \end{cases} \tag{6.2}$$

where $b_e = (|\Omega_e^\varepsilon|)^{-1} \int_{\Omega_e^\varepsilon} W_{T_N^\varepsilon}(p_e^\varepsilon, \mathbf{s})\, \mathrm{d}\mathbf{s}$, $p_e^\varepsilon = \partial \overline{V}^\varepsilon(v) \cap e$ is the end point of $\overline{V}^\varepsilon(v)$ corresponding to $e \in N(v)$, and $\{\psi_{(e)}\}$ is the partition of unity in a neighborhood of the vertex v defined in Section 3. The functions ϕ_Q^ε and ϕ_P^ε converge to the identity function as ε tends to zero.

6.1. **Rayleigh quotients of Schrödinger operator on T_1 and T_N^ε.** The comparison between the Rayleigh quotients on T_1 and T_N^ε involves the construction of transformations $Q^\varepsilon : H^1_{0,\rho^*}(T_1) \to H^1_0(T_N^\varepsilon)$ and $P^\varepsilon : H^1_0(T_N^\varepsilon) \to H^1_{0,\rho^*}(T_1)$, where $\rho^* : T_1 \to \mathbb{R}$ is defined by $\rho^*(\theta) := \delta_e^{(N-1)}|\Omega_e|$ for $\theta \in e$. We devote the following two subsections for the definitions of these transformations.

6.1.1. *The mapping $Q^\varepsilon : H^1_{0,\rho^*}(T_1) \to H^1_0(T_N^\varepsilon)$.* Given a function $f \in H^1_{\rho^*}(T_1)$ and a vertex v, we denote by $f_e^\varepsilon = f(p_e^\varepsilon)$ and $\vec{f}^\varepsilon = \{f_e^\varepsilon\}_{e\in N(v)}$. $Q^\varepsilon(f)$ is defined as follows:

$$Q^\varepsilon(f)(\mathbf{x}) = \begin{cases} f(\theta) & \mathbf{x} = (\theta, s) \in E^\varepsilon, \\ \sum_{e\in N(v)} f_e^\varepsilon \phi_{(e)}^\varepsilon & \mathbf{x} \in V^\varepsilon(v), \end{cases} \tag{6.3}$$

where $\{\phi_{(e)}^\varepsilon\}$ is a partition of unity of $V^\varepsilon(v)$ as defined in Section 3. We denote $Q(f) := Q^1(f)$. We also define

$$\rho_Q^\varepsilon(\theta) = \begin{cases} \rho^* & \theta \in \overline{E}^\varepsilon(e), \\ \max\left\{\frac{\alpha^A}{\beta^{\overline{A}}}, \frac{\alpha^B}{\beta^{\overline{B}}}\right\}\rho^* & \theta \in \overline{V}^\varepsilon(v), \end{cases} \tag{6.4}$$

where $\alpha^A, \beta^{\overline{A}}, \alpha^B$ and $\beta^{\overline{B}}$ are defined in Section 3 and Lemma 3.1.

Lemma 6.1. *There exists $c > 0$ such that for any $f \in H^1_{0,\rho^*}(T_1)$ and $0 < \varepsilon < 1$, we have*

(1) $Q^\varepsilon(f) \in L^2(T_N^\varepsilon)$.

(2) $\int_{T_N^\varepsilon} |\nabla Q^\varepsilon(f)|^2\, \mathrm{d}\mathbf{x} \le \varepsilon^{(N-1)} \int_{T_1} |f'|^2 \rho_Q^\varepsilon\, \mathrm{d}\theta$. *Moreover,* $Q^\varepsilon(f) \in H^1_0(T_N^\varepsilon)$.

(3) $\int_{T_N^\varepsilon} |Q^\varepsilon(f)|^2 \mathrm{d}\mathbf{x} \ge \varepsilon^{(N-1)} \int_{T_1} (|f|^2 - c\varepsilon|f'|^2)\rho^* \mathrm{d}\theta$.

(4) $\int_{T_N^\varepsilon} W_{T_n^\varepsilon} |Q^\varepsilon(f)|^2\, \mathrm{d}\mathbf{x} \le \varepsilon^{(N-1)} \int_{T_1} (W_{T_1,\varepsilon}|f|^2 + O(\varepsilon)|f'|^2)\rho^*\, \mathrm{d}\theta$.

Proof. 1. We denote a normalized connector $\overline{V} := (\varepsilon\delta)^{-1}\overline{V}^\varepsilon(v)$, where $\delta = \delta_v = \delta^{\mathrm{gen}(v)}$ corresponds to the vertex v in question, and

$$\widehat{f}(\theta) := f\left(\frac{\theta}{\varepsilon\delta}\right), \qquad \widehat{\rho^*}(\theta) := \rho^*\left(\frac{\theta}{\varepsilon\delta}\right) \tag{6.5}$$

are the representation of f and ρ^* in $\overline{V}$ (here $\theta = 0$ corresponds to v).

$$\int_{E^\varepsilon} |Q^\varepsilon(f)|^2\, \mathrm{d}\mathbf{x} = \int_{\overline{E}^\varepsilon}\int_{\Omega^\varepsilon} |f(\theta)|^2 \mathrm{d}s\mathrm{d}\theta = \int_{\overline{E}^\varepsilon} (\varepsilon\delta)^{(N-1)}|\Omega^\varepsilon||f|^2\, \mathrm{d}\theta = \varepsilon^{(N-1)} \int_{\overline{E}^\varepsilon} |f|^2\rho^*\, \mathrm{d}\theta.$$

For the connector V^ε, we have by Lemma 3.1 that

$$\int_{V^\varepsilon} |Q^\varepsilon(f)|^2\, \mathrm{d}\mathbf{x} = (\varepsilon\delta)^N \int_V |Q(f)|^2\, \mathrm{d}\mathbf{x} \le (\varepsilon\delta)^N \alpha^B |\vec{f}|^2.$$

By Lemma 3.3 and since $\delta < 1$, we have that

$$(\varepsilon\delta)^N\alpha^B|\vec{f}|^2 \le \frac{(\varepsilon\delta)^N\alpha^B}{\delta^{(N-1)}\beta^{\overline{B}}} \int_{\overline{V}} (|\widehat{f}'|^2 + |\widehat{f}|^2)\widehat{\rho^*}\, \mathrm{d}\theta \le \frac{\varepsilon^{(N-1)}\alpha^B}{\beta^{\overline{B}}} \int_{\overline{V}^\varepsilon} (|\varepsilon f'|^2 + |f|^2)\rho^*\, \mathrm{d}\theta.$$

In particular, we proved

$$\|Q^\varepsilon(f)\|^2_{L^2(T_N^\varepsilon)} \le \varepsilon^{(N+1)}\|f'\|^2_{L^2_{\rho_Q^\varepsilon}(T_1)} + \varepsilon^{(N-1)}\|f\|^2_{L^2_{\rho_Q^\varepsilon}(T_1)}. \tag{6.6}$$

2.

$$\int_{E^\varepsilon} |\nabla Q^\varepsilon(f)|^2\, \mathrm{d}\mathbf{x} = \int_{E^\varepsilon} |f'|^2\, \mathrm{d}\mathbf{x} = \int_{\overline{E}^\varepsilon}\int_{\Omega^\varepsilon} |f'|^2 \mathrm{d}s\mathrm{d}\theta = \varepsilon^{(N-1)} \int_{\overline{E}^\varepsilon} |f'|^2\rho^*\, \mathrm{d}\theta.$$

For V^ε we use similar considerations to those used in part 1:

$$\int_{V^\varepsilon} |\nabla Q^\varepsilon(f)|^2 \,\mathrm{d}\mathbf{x} = (\varepsilon\delta)^{(N-2)} \int_V |\nabla Q(f)|^2 \,\mathrm{d}\mathbf{x} = (\varepsilon\delta)^{(N-2)} \vec{f} A \vec{f}^*$$
$$\leq (\varepsilon\delta)^{(N-2)} \alpha^A |\vec{f} \llcorner \vec{1}|^2 \leq \frac{(\varepsilon\delta)^{(N-2)}}{\delta^{(N-1)}} \frac{\alpha^A}{\beta^{\overline{A}}} \int_{\overline{V}} |\widehat{f}'|^2 \rho^* \mathrm{d}\theta = \varepsilon^{(N-1)} \frac{\alpha^A}{\beta^{\overline{A}}} \int_{\overline{V}^\varepsilon} |f'|^2 \rho^* \mathrm{d}\theta.$$

So, $Q^\varepsilon(f) \in H_0^1(T_N^\varepsilon)$ by definition (6.4).

3. By Lemma 4.10,

$$\int_{T_N^\varepsilon} |Q^\varepsilon(f)|^2 \,\mathrm{d}\mathbf{x} \geq \int_{T_N^\varepsilon \setminus \cup_v V^\varepsilon(v)} |Q^\varepsilon(f)|^2 \,\mathrm{d}\mathbf{x} = \varepsilon^{(N-1)} \int_{\cup_e \overline{E}(e)} |f|^2 \rho^* \,\mathrm{d}\theta$$
$$= \varepsilon^{(N-1)} \int_{T_1} |f|^2 \rho^* \,\mathrm{d}\theta - \varepsilon^{(N-1)} \int_{\cup_v \overline{V}^\varepsilon(v)} |f|^2 \rho^* \mathrm{d}\theta \geq \varepsilon^{(N-1)} \int_{T_1} |f|^2 \rho^* \mathrm{d}\theta - c\varepsilon^N \int_{T_1} |f'|^2 \rho^* \mathrm{d}\theta.$$

The proof of (4) is a simple extension of (6.6). □

Corollary 6.2. *There exists a constant $c > 0$ such that for all $f \in H^1_{0,\rho^*}(T_1)$ and $0 < \varepsilon$ sufficiently small, the Rayleigh quotients*

$$R_{H_0^1(T_N^\varepsilon)}[Q^\varepsilon f] := \frac{\int_{T_N^\varepsilon} (|\nabla(Q^\varepsilon f)|^2 + W_{T_n^\varepsilon} |Q^\varepsilon(f)|^2) \,\mathrm{d}\mathbf{x}}{\int_{T_N^\varepsilon} |Q^\varepsilon f|^2 \,\mathrm{d}\mathbf{x}},$$

and

$$R_{H^1_{0,\rho^*}(T_1)}[f] := \frac{\int_{T_1} (|f_\theta|^2 \rho_Q^\varepsilon + W_{T_1,\varepsilon} |f|^2 \rho^*) \,\mathrm{d}\theta}{\int_{T_1} |f|^2 \rho^* \,\mathrm{d}\theta}$$

satisfy the inequality

$$R_{H_0^1(T_N^\varepsilon)}[Q^\varepsilon f] \leq \frac{(1 + O(\varepsilon)) R_{H^1_{0,\rho^*}(T_1)}[f]}{1 - c\varepsilon R_{H^1_{0,\rho^*}(T_1)}[f]}. \tag{6.7}$$

Remark 6.3. Notice that $R_{H^1_{0,\rho^*}(T_1)}[f]$ depends on ε and is the Rayleigh quotient of the width-weighted operator A_ε defined in (6.1), substituting $\rho_{\alpha,\varepsilon} = \rho_Q^\varepsilon$ and $\rho_{b,\varepsilon} = \rho^*$.

6.1.2. *The mapping $P^\varepsilon : H_0^1(T_N^\varepsilon) \to H^1_{0,\rho^*}(T_1)$.* Given a function $u \in H_0^1(T_N^\varepsilon)$, a vertex v and edges $e \in N(v)$, we denote

$$u_e := \frac{1}{|\Omega_e^\varepsilon|} \int_{\Omega_e^\varepsilon} u(p_e^\varepsilon, \mathbf{s}) \,\mathrm{d}\mathbf{s}, \qquad \vec{u} = \vec{u}_v := \{u_e\}_{e \in V(v)},$$

where $p_e^\varepsilon = \partial \overline{V}^\varepsilon(v) \cap e$ are the end points of $\overline{V}^\varepsilon(v)$. Define

$$P^\varepsilon(u)(\theta) := \begin{cases} \dfrac{\int_{\Omega^\varepsilon} u(\theta, \mathbf{s}) \mathrm{d}\mathbf{s}}{|\Omega^\varepsilon|} & \theta \in \overline{E}^\varepsilon, \\ \displaystyle\sum_{e \in V(v)} u_e \psi^\varepsilon_{(e)}(\theta) & \theta \in \overline{V}^\varepsilon, \end{cases} \tag{6.8}$$

where $\{\psi_{(e)}\}$ is the partition of unity in a neighborhood of the vertex v defined in Section 3. We also define

$$\rho_P^\varepsilon(\theta) := \begin{cases} \rho^* & \theta \in \overline{E}^\varepsilon, \\ \min\left\{\frac{\beta^A}{\alpha^{\overline{A}}}, \frac{\beta^B}{\alpha^{\overline{B}}}\right\} \rho^* & \theta \in \overline{V}^\varepsilon. \end{cases} \tag{6.9}$$

Lemma 6.4. *There exists $c >$ such that for any $u \in H_0^1(T_N^\varepsilon)$ and $0 < \varepsilon$ sufficiently small, we have*

(1) $P^\varepsilon(u) \in L^2_{\rho^*}(T_1)$.

(2) $\varepsilon^{(N-1)} \int_{T_1} |(P^\varepsilon u)'|^2 \rho^\varepsilon_P \,\mathrm{d}\theta \le \int_{T^\varepsilon_N} |\nabla u|^2 \,\mathrm{d}\mathbf{x}$. *In particular,* $P^\varepsilon(u) \in H^1_{0,\rho^*}(T_1)$.

(3) $\varepsilon^{(N-1)} \int_{T_1} |P^\varepsilon u|^2 \rho^* \,\mathrm{d}\theta \ge \int_{T^\varepsilon_N} [(1-\sqrt{\varepsilon})|u|^2 - c\varepsilon|\nabla u|^2] \,\mathrm{d}\mathbf{x}$.

(4) $\varepsilon^{(N-1)} \int_{T_1} W_{T_1,\varepsilon} |P^\varepsilon u|^2 \rho^* \,\mathrm{d}\theta \le \int_{T^\varepsilon_N} (1+2\sqrt{\varepsilon})(W_{T^\varepsilon_N}|u|^2 + O(\varepsilon)|\nabla u|^2) \,\mathrm{d}\mathbf{x}$.

Proof. Throughout the proof we denote $\widehat{u}(\mathbf{x}) = u(\delta\varepsilon\mathbf{x})$ for $\mathbf{x} \in V$ (so, $\varepsilon\delta\mathbf{x} \in V^\varepsilon$). Similarly, $\widetilde{u}(\theta, \mathbf{s}) = u(\theta, \varepsilon\delta\mathbf{s})$ for $(\theta, \mathbf{s}) \in \overline{E} \times \Omega$ (so, $(\theta, \varepsilon\delta\mathbf{s}) \in E^\varepsilon$).

1. For each edge E^ε,

$$\varepsilon^{(N-1)} \int_{\overline{E}^\varepsilon} |P^\varepsilon u|^2 \rho^* \,\mathrm{d}\theta = \int_{\overline{E}^\varepsilon} |\Omega^\varepsilon||P^\varepsilon u|^2 \,\mathrm{d}\theta = \int_{\overline{E}^\varepsilon} |\Omega^\varepsilon| \left| \frac{1}{|\Omega^\varepsilon|} \int_{\Omega^\varepsilon} u(\theta, \mathbf{s}) \mathrm{d}\mathbf{s} \right|^2 \mathrm{d}\theta$$
$$\le \int_{\overline{E}^\varepsilon} \int_{\Omega^\varepsilon} |u(\theta, \mathbf{s})|^2 \mathrm{d}\mathbf{s}\mathrm{d}\theta = \int_{E^\varepsilon} |u(\theta, \mathbf{s})|^2 \mathrm{d}\mathbf{x}.$$

Using Lemma 3.1, we obtain for the connector V^ε

$$\varepsilon^{(N-1)} \int_{\overline{V}^\varepsilon} |P^\varepsilon u|^2 \rho^* \mathrm{d}\theta = \varepsilon^N \delta \sum_{e,\tilde{e} \in N(v)} u_e \overline{u_{\tilde{e}}} \int_{\overline{V}} \psi_{(e)} \psi_{(\tilde{e})} \rho^* \mathrm{d}\theta = \varepsilon^N \delta \vec{u} \overline{B} (\vec{u})^* \le (\varepsilon\delta)^N \alpha^{\overline{B}} |\vec{u}|^2.$$

By Lemma 3.5, and assuming $\delta < 1$, we obtain

$$(\varepsilon\delta)^N \alpha^{\overline{B}} |\vec{u}|^2 \le (\varepsilon\delta)^N \frac{\alpha^{\overline{B}}}{\beta^B} \int_V (|\nabla \widehat{u}|^2 + |\widehat{u}|^2) \,\mathrm{d}\mathbf{x} \le \frac{\alpha^{\overline{B}}}{\beta^B} \int_{V^\varepsilon} (|\varepsilon \nabla u|^2 + |u|^2) \,\mathrm{d}\mathbf{x}.$$

2. For an edge E^ε we have

$$\varepsilon^{(N-1)} \int_{\overline{E}^\varepsilon} \left|(P^\varepsilon u)'\right|^2 \rho^* \,\mathrm{d}\theta = \varepsilon^{(N-1)} \int_{\overline{E}^\varepsilon} \left| \frac{1}{|\Omega^\varepsilon|} \int_{\Omega^\varepsilon} \frac{\partial u(\theta, s)}{\partial \theta} \mathrm{d}\mathbf{s} \right|^2 \rho^* \mathrm{d}\theta \tag{6.10}$$
$$= \int_{\overline{E}^\varepsilon} |\Omega^\varepsilon|^{-1} \left(\int_{\Omega^\varepsilon} \frac{\partial u(\theta, s)}{\partial \theta} \mathrm{d}\mathbf{s} \right)^2 \mathrm{d}\theta \le \int_{E^\varepsilon} |\nabla u|^2 \,\mathrm{d}\mathbf{x}.$$

For V^ε, we have by Lemma 3.1 and (6.8)

$$\varepsilon^{(N-1)} \int_{\overline{V}^\varepsilon} \left|(P^\varepsilon u)'\right|^2 \rho^* \,\mathrm{d}\theta = \varepsilon^{(N-1)} \sum_{e,\tilde{e} \in N(v)} u_e \overline{u_{(\tilde{e})}} \int_{\overline{V}^\varepsilon} (\psi^\varepsilon_{(e)})' (\psi^\varepsilon_{(\tilde{e})})' \rho^* \,\mathrm{d}\theta \tag{6.11}$$
$$= \varepsilon^{(N-2)} \delta^{-1} \vec{u} \overline{\mathbf{A}} (\vec{u})^* \le (\varepsilon\delta)^{(N-2)} \alpha^{\overline{A}} |\vec{u} \llcorner \vec{1}|^2$$
$$\le (\varepsilon\delta)^{(N-2)} \frac{\alpha^{\overline{A}}}{\beta^A} \int_V |\nabla \widehat{u}|^2 \,\mathrm{d}\mathbf{x} = \frac{\alpha^{\overline{A}}}{\beta^A} \int_{V^\varepsilon} |\nabla u|^2 \,\mathrm{d}\mathbf{x}.$$

3. In the edges E^ε, we use the same argument as in [21]. By the inequality

$$(a+b)^2 \ge (1-\sqrt{\varepsilon})a^2 - b^2/\sqrt{\varepsilon}, \tag{6.12}$$

we have that

$$\varepsilon^{(N-1)}\int_{\overline{E}^\varepsilon}|P^\varepsilon u|^2\rho^*\,\mathrm{d}\theta=\int_{E^\varepsilon}|P^\varepsilon u|^2\,\mathrm{d}\mathbf{s}\,\mathrm{d}\theta=\int_{E^\varepsilon}|u+(P^\varepsilon u-u)|^2\,\mathrm{d}\mathbf{s}\,\mathrm{d}\theta$$
$$\geq\int_{\overline{E}^\varepsilon}\int_{\Omega^\varepsilon}\left[(1-\sqrt{\varepsilon})|u(\theta,\mathbf{s})|^2-\frac{1}{\sqrt{\varepsilon}}|P^\varepsilon u-u(\theta,\mathbf{s})|^2\right]\mathrm{d}\mathbf{s}\mathrm{d}\theta$$
$$=\varepsilon^{(N-1)}\int_{\overline{E}^\varepsilon}\int_{\Omega}\left[(1-\sqrt{\varepsilon})|\widetilde{u}(\theta,\mathbf{s}|^2)-\frac{1}{\sqrt{\varepsilon}}|P\widetilde{u}-\widetilde{u}(\theta,\mathbf{s})|^2\right]\mathrm{d}\mathbf{s}\mathrm{d}\theta.$$

Notice that for each θ we have that $P\widetilde{u}(\theta)-\widetilde{u}(\theta,\mathbf{s})$ has average zero on Ω. By Poincaré inequality in $H^1(\Omega)$, there exists a constant $D>0$ such that $\int_\Omega|P\widetilde{u}-\widetilde{u}|^2\,\mathrm{d}\mathbf{s}\leq D\int_\Omega|\nabla\widetilde{u}|^2\,\mathrm{d}\mathbf{s}$ and hence,

$$\varepsilon^{(N-1)}\int_{\overline{E}^\varepsilon}|P^\varepsilon u|^2\rho^*\mathrm{d}\theta\geq\varepsilon^{(N-1)}\int_{\overline{E}^\varepsilon}\int_{\Omega}\left[(1-\sqrt{\varepsilon})|\widetilde{u}(\theta,\mathbf{s})|^2-\frac{1}{\sqrt{\varepsilon}}D|\nabla\widetilde{u}(\theta,\mathbf{s})|^2\right]\mathrm{d}\mathbf{s}\mathrm{d}\theta$$
$$=\int_{E^\varepsilon}\left[(1-\sqrt{\varepsilon})|u|^2-\varepsilon^{3/2}D|\nabla u|^2\right]\,\mathrm{d}\mathbf{x}.$$

Therefore,

$$\int_{\cup_e E^\varepsilon(e)}[(1-\sqrt{\varepsilon})|u|^2-\varepsilon^{3/2}D|\nabla u|^2]\,\mathrm{d}\mathbf{x}\leq\varepsilon^{(N-1)}\int_{T_1}\rho^*|P^\varepsilon u|^2\,\mathrm{d}\theta.$$

On the other hand, by Lemma 4.11,

$$\int_{\cup_v V^\varepsilon(v)}[(1-\sqrt{\varepsilon})|u|^2-\varepsilon^{3/2}D|\nabla u|^2]\mathrm{d}\mathbf{x}\leq\int_{\cup_v V^\varepsilon(v)}(1-\sqrt{\varepsilon})|u|^2\mathrm{d}\mathbf{x}\leq c\varepsilon(1-\sqrt{\varepsilon})\int_{T_N^\varepsilon}|\nabla u|^2\mathrm{d}\mathbf{x}.$$

Summing the last two inequalities, we obtain the proof of part 3.

4. Since

$$\varepsilon^{(N-1)}\int_{\overline{E}^\varepsilon}W_{T_1,\varepsilon}|P^\varepsilon u|^2\rho^*\,\mathrm{d}\theta=\int_{E^\varepsilon}W_{T_N^\varepsilon}|P^\varepsilon u|^2\,\mathrm{d}\mathbf{s}\,\mathrm{d}\theta,$$

it is sufficient to prove for the edges that

$$\int_{E^\varepsilon}W_{T_N^\varepsilon}|P^\varepsilon u|^2\,\mathrm{d}\mathbf{s}\,\mathrm{d}\theta\leq\int_{E^\varepsilon}(1+2\sqrt{\varepsilon})W_{T_N^\varepsilon}|u|^2+O(\varepsilon)|\nabla u|^2)\,\mathrm{d}\mathbf{x}.$$

Using (6.12), we have that

$$\int_{E^\varepsilon}W_{T_N^\varepsilon}|u|^2\,\mathrm{d}\mathbf{x}\geq(1-\sqrt{\varepsilon})\int_{E^\varepsilon}W_{T_N^\varepsilon}|P^\varepsilon u|^2\,\mathrm{d}\mathbf{s}\,\mathrm{d}\theta-\frac{1}{\sqrt{\varepsilon}}\int_{E^\varepsilon}W_{T_N^\varepsilon}|u-P^\varepsilon u|^2\,\mathrm{d}\mathbf{s}\,\mathrm{d}\theta.$$

Therefore, if $0<\varepsilon<1$ is small enough so that $1\leq(1-\sqrt{\varepsilon})(1+2\sqrt{\varepsilon})$, then by Poincaré inequality, there exists a constant D such that

(6.13)
$$\int_{E^\varepsilon}W_{T_N^\varepsilon}|P^\varepsilon u|^2\,\mathrm{d}\mathbf{s}\,\mathrm{d}\theta\leq(1+2\sqrt{\varepsilon})\left\{\int_{E^\varepsilon}W_{T_N^\varepsilon}|u|^2\,\mathrm{d}\mathbf{x}+\frac{1}{\sqrt{\varepsilon}}\int_{E^\varepsilon}W_{T_N^\varepsilon}|u-P^\varepsilon u|^2\right\}\,\mathrm{d}\mathbf{s}\,\mathrm{d}\theta$$
$$\leq(1+2\sqrt{\varepsilon})\int_{E^\varepsilon}W_{T_N^\varepsilon}|u|^2\,\mathrm{d}\mathbf{x}+C_W(1+2\sqrt{\varepsilon})\frac{1}{\sqrt{\varepsilon}}\int_{\overline{E}^\varepsilon}\int_\Omega|\widetilde{u}-P\widetilde{u}|^2(\varepsilon\delta)^{(N-1)}\,\mathrm{d}\mathbf{s}\,\mathrm{d}\theta$$
$$\leq(1+2\sqrt{\varepsilon})\int_{E^\varepsilon}W_{T_N^\varepsilon}|u|^2\,\mathrm{d}\mathbf{x}+C_WD(1+2\sqrt{\varepsilon})\frac{(\varepsilon\delta)^2}{\sqrt{\varepsilon}}\int_{E^\varepsilon}|\nabla_{\mathbf{s}}u|^2\,\mathrm{d}\mathbf{s}\,\mathrm{d}\theta$$
$$\leq(1+2\sqrt{\varepsilon})\int_{E^\varepsilon}W_{T_N^\varepsilon}|u|^2\,\mathrm{d}\mathbf{x}+O(\varepsilon^{3/2})\int_{E^\varepsilon}|\nabla u|^2\,\mathrm{d}\mathbf{x}.$$

For the connectors we obtain by Lemma 4.10 and part 2,

$$\varepsilon^{(N-1)}\int_{\cup_v \overline{V}^\varepsilon(v)} W_{T_1,\varepsilon}|P^\varepsilon u|^2\rho^*\,\mathrm{d}\theta \leq \varepsilon^N C_W\int_{T_1}\left|(P^\varepsilon u)'\right|^2\rho^*\,\mathrm{d}\theta \leq \varepsilon C_W\int_{T_N^\varepsilon}|\nabla u|^2\,\mathrm{d}\mathbf{x}$$

which, together with (6.13), yields the proof of part 4. □

Corollary 6.5. *For all $\varepsilon > 0$ sufficiently small, there exists a constant $c > 0$ such that the Rayleigh quotients*

$$R_{H^1_{0,\rho^*}(T_1)}[P^\varepsilon u] := \frac{\int_{T_1}\left(|(P^\varepsilon u)'|^2\rho_P^\varepsilon + W_{T_1,\varepsilon}|P^\varepsilon u|^2\rho^*\right)\mathrm{d}\theta}{\int_{T_1}|u|^2\rho^*\mathrm{d}\theta},$$

and

$$R_{H^1_0(T_N^\varepsilon)}[u] := \frac{\int_{T_N^\varepsilon}(|\nabla u|^2 + W_{T_n^\varepsilon}|u|^2)\mathrm{d}\mathbf{x}}{\int_{T_N^\varepsilon}|u|^2\mathrm{d}\mathbf{x}}$$

satisfy

$$R_{H^1_{0,\rho^*}(T_1)}[P^\varepsilon u] \leq \frac{[1+O(\sqrt{\varepsilon})]R_{H^1_0(T_N^\varepsilon)}[u]}{1-\sqrt{\varepsilon}-c\varepsilon R_{H^1_0(T_N^\varepsilon)}[u]} \qquad \forall u\in H^1_0(T_N^\varepsilon). \tag{6.14}$$

Remark 6.6. Notice that $R_{H^1_{0,\rho^*}(T_1)}[P^\varepsilon u]$ is the Rayleigh quotient of the width-weighted operator A_ε defined in (6.1), substituting $\rho_{\alpha,\varepsilon} = \rho_P^\varepsilon$ and $\rho_{b,\varepsilon} = \rho^*$.

6.2. T_1**-based estimates for the spectrum on** T_N^ε**.** Rubinstein and Schatzman have proved the following general lemma [21].

Lemma 6.7. *Let A_j be bounded below, selfadjoint operators defined on Hilbert spaces H_j, where $j = 0,1$, and let $\{\lambda_m(A_j)\}$ be the nondecreasing sequence of the corresponding eigenvalues. Denote by D_j the domain of the maximal quadratic form associated with A_j and by R_j the Rayleigh quotient associated with A_j. Suppose that there exists a continuous linear operator S mapping D_1 to D_0 and an increasing function $\phi : \mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ such that* $\exp(-\phi)$ *is continuous, and*

$$R_0(Su) \leq \phi(R_1(u)) \qquad \forall u\in D_1\setminus\ker(S).$$

Assume that for a given m,

$$\mu := \inf\{R_1(v)\mid v\in D_1\cap\ker(S), v\neq 0\} > \lambda_m(A_1). \tag{6.15}$$

Then

$$\lambda_m(A_0) \leq \phi(\lambda_m(A_1)). \tag{6.16}$$

Using Lemma 6.7, we obtain bounds for the eigenvalues of T_N^ε. Let ν_m^ε denotes the m-th eigenvalue of the Schrödinger operator

$$L_\varepsilon := -\Delta + W_{T_N^\varepsilon}.$$

Denote the operators

$$A_Q^\varepsilon := -\frac{1}{\rho^*}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\rho_Q^\varepsilon\frac{\mathrm{d}}{\mathrm{d}\theta}\right) + W_{T_1,\varepsilon}, \qquad A_P^\varepsilon := -\frac{1}{\rho^*}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\rho_P^\varepsilon\frac{\mathrm{d}}{\mathrm{d}\theta}\right) + W_{T_1,\varepsilon},$$

and let μ_m^ε (resp. λ_m^ε) be the m-th eigenvalue of A_Q^ε (resp. A_P^ε). We will omit the superscript ε in ν_m^ε, μ_m^ε, and λ_m^ε whenever there is no danger of confusion.

Theorem 6.8. *Using the notations above, for all $M \in \mathbb{N}$ there exist $\varepsilon_M > 0$ and a constant $c > 0$ such that for all $m \leq M$ and $0 < \varepsilon < \varepsilon_M$, we have*

$$\nu_m^\varepsilon \leq \phi_Q^\varepsilon(\mu_m^\varepsilon), \tag{6.17}$$

and

$$\lambda_m^\varepsilon \leq \phi_P^\varepsilon(\nu_m^\varepsilon), \tag{6.18}$$

where

$$\phi_Q^\varepsilon(x) := \begin{cases} \dfrac{(1+c\varepsilon)x}{1-c\varepsilon x} & x < (c\varepsilon)^{-1}, \\ +\infty & \text{otherwise}, \end{cases} \qquad \text{and} \quad \phi_P^\varepsilon(x) := \begin{cases} \dfrac{(1+c\varepsilon)x}{1-\sqrt{\varepsilon}-c\varepsilon x} & x < \dfrac{1-\sqrt{\varepsilon}}{c\varepsilon}, \\ +\infty & \text{otherwise}. \end{cases}$$

Proof of Theorem 6.8. Without loss of generality, we assume that W_{T_1} is positive. In order to prove (6.17), we wish to apply Lemma 6.7 on $S = Q^\varepsilon, D_0 = H_0^1(T_N^\varepsilon)$, $D_1 = H^1_{0,\rho^*}(T_1), A_0 = L_\varepsilon, A_1 = A_Q^\varepsilon$, $R_0 = R_{H_0^1(T_N^\varepsilon)}$, and $R_1 = R_{H^1_{0,\rho^*}(T_1)}$. We, therefore, show that there exists $C > 0$ such that for any $\varepsilon > 0$

$$\inf\left\{R_{H^1_{0,\rho^*}(T_1)}[f] \mid f \in \ker Q^\varepsilon,\ f \neq 0\right\} \geq \frac{1}{C\varepsilon^2}\,. \tag{6.19}$$

Indeed,

$$\ker Q^\varepsilon = \left\{f \in H^1_{0,\rho^*}(T_1) \mid f(\theta) = 0 \quad \forall \theta \in T_1 \setminus \cup_v \overline{V}^\varepsilon\right\}.$$

Therefore, in order to estimate $R_{H^1_{0,\rho^*}(T_1)}[f]$ for $f \in \ker(Q^\varepsilon)$, we actually need to estimate this quotient in each component $\overline{V}^\varepsilon \cap e$. However, we have that

$$|f(\theta)|^2 = \left|\int_{p^\varepsilon}^\theta f' \mathrm{d}\vartheta\right|^2 \leq |p^\varepsilon - \theta| \int_{p^\varepsilon}^\theta |f'|^2 \mathrm{d}\vartheta,$$

where $p^\varepsilon \in \partial \overline{V}^\varepsilon$. Multiply the above by ρ^* (which, we recall, is constant on each component $\overline{V}^\varepsilon \cap e$), we find the existence of $C > 0$ such that

$$\int_{\overline{V}^\varepsilon \cap e} |f|^2 \rho^* \mathrm{d}\theta \leq \int_{\overline{V}^\varepsilon \cap e} \left(|p^\varepsilon - \theta| \int_{p^\varepsilon}^\theta |f'|^2 \rho^* \mathrm{d}\vartheta\right) \mathrm{d}\theta \leq C\varepsilon^2 \int_{\overline{V} \cap e} |f'|^2 \rho^*\, \mathrm{d}\theta.$$

Thus, (6.19) is verified provided ε is sufficiently large. Hence, (6.17) follows from Corollary 6.2 and Lemma 6.7.

In order to prove (6.18), we wish to apply Lemma 6.7 to $S = P^\varepsilon, D_0 = H^1_{0,\rho^*}(T_1)$, $D_1 = H_0^1(T_N^\varepsilon), A_0 = A_P^\varepsilon, A_1 = L_\varepsilon, R_0 = R_{H^1_{0,\rho^*}(T_1)}$, and $R_1 = R_{H_0^1(T_N^\varepsilon)}$. To this end, we show that there exists $C > 0$ such that for any $\varepsilon > 0$

$$\inf\left\{R_{H_0^1(T_N^\varepsilon)}[u] \ \mid\ u \in \ker P^\varepsilon,\ u \neq 0\right\} \geq \frac{1}{C\varepsilon}. \tag{6.20}$$

We notice that if $u \in \ker P^\varepsilon$, then its averages on the cross sections Ω_j^ε of E_j^ε vanish. Therefore, using the $(N-1)$-dimensional Poincaré inequality for functions whose average is zero, we obtain that there is a constant D such that:

$$\int_{E^\varepsilon} |u|^2 \,\mathrm{d}\mathbf{s}\,\mathrm{d}\theta \leq D\varepsilon^{2(N-1)} \int_{\overline{E}^\varepsilon}\int_{\Omega^\varepsilon} |\nabla_s u|^2 \,\mathrm{d}\mathbf{s}\,\mathrm{d}\theta \leq D\varepsilon^{2(N-1)} \int_{E^\varepsilon} (|\nabla u|^2 + W_{T_N^\varepsilon}|u|^2) \,\mathrm{d}\mathbf{s}\,\mathrm{d}\theta. \tag{6.21}$$

By Lemma 4.11, there exists $C > 0$ such that for any $u \in H_0^1(T_N^\varepsilon)$

$$\int_{\cup_v V^\varepsilon(v)} |u|^2 \,\mathrm{d}\mathbf{x} \leq C\varepsilon \int_{T_N^\varepsilon} (|\nabla u|^2 + W_{T_N^\varepsilon}|u|^2) \,\mathrm{d}\mathbf{x}. \tag{6.22}$$

Therefore, (6.21) and (6.22) imply (6.20). Thus, (6.18) follows by Corollary 6.5 and Lemma 6.7. $\square$

Remark 6.9. Theorem 6.8 is similar to [21, Theorem5] proved for a finite graph with a constant-width thin domain.

Theorem 6.10. *For each $m \in \mathbb{N}$, the m-the eigenvalue of the Schrödinger operator L_ε on $H_0^1(T_N^\varepsilon)$ converges as $\varepsilon \to 0$ to the m-the eigenvalue of limit width-weighted operator A on $H_0^1(T_1)$.*

Proof. We use in this proof the notations of Theorem 6.8. Notice that for small enough ε, ϕ_Q^ε and ϕ_P^ε are continuous monotone increasing function, which satisfy

$$\lim_{\varepsilon\to 0} \phi_Q^\varepsilon(x) = x, \quad \lim_{\varepsilon\to 0} \phi_P^\varepsilon(x) = x.$$

Moreover, since the operators we refer to in Theorem 6.8 satisfy the conditions of Theorem 5.4, we have for each $m \in \mathbb{N}$ that both μ_m^ε and λ_m^ε (see (6.17) and (6.18)) converge as $\varepsilon \to 0$ to the m-th eigenvalue of the limit width-weighted operator A. Since A has a discrete spectrum, the result follows. □

7. Convergence of eigenfunctions of Laplace operator on T_N^ε

In [7, 8], Kosugi has proved that the solutions of $\Delta u + f(u) = 0$ in thin network-shaped bounded domains that satisfy Neumann boundary condition, converge to solutions of appropriate equations on the skeleton of the domain. In [7], Kosugi deals only with domains which are formed by joining straight tubes around some graph, while in [8] the results are extended to general domains around graphs. However, trees with infinite number of vertices and nonsmooth boundaries are not considered in these papers. Using the transformation P^ε developed for Theorem 6.5, we give a simple proof for the convergence of projections into $H_{0,*}^1(T_1)$ of eigenfunctions u_ε of the Laplace operator on $H_0^1(T_N^\varepsilon)$. Specifically, we show in Theorem 7.2 that $P^\varepsilon u_\varepsilon$ converges to eigenfunctions of the following limit width-weighted operator on T_1

$$L_* u := (\rho^*)^{-1}\left(\rho^* u'\right)'.$$

First, we need to prove the following auxiliary Lemma.

Lemma 7.1. *Assume that $u \in H_0^1(T_n^\varepsilon)$ satisfies $||u||_{H_0^1(T_n^\varepsilon)} = \varepsilon^{(n-1)/2}$. Fix a vertex v, and denote by p_e the 'end point' in $\overline{V}^\varepsilon \cap \overline{E}^\varepsilon(e)$. Then there is a constant C which depends on v but is independent on ε such that for $e, \tilde{e} \in N(v)$ we have*

$$|P^\varepsilon u(p_e) - P^\varepsilon u(p_{\tilde{e}})| \le C\sqrt{\mathrm{dist}(p_e, p_{\tilde{e}})}\,, \tag{7.1}$$

where $\mathrm{dist}(\cdot,\cdot)$ *is the standard distance function on* T_1.

Proof. Notice that since $C^1(T_n^\varepsilon)$ is dense in $H_0^1(T_n^\varepsilon)$ we may assume without loss of generality that $u \in C^1(T_n^\varepsilon)$.
Let $q, r \in \overline{V}^\varepsilon \cap e$. By (6.11),

$$|P^\varepsilon u(q) - P^\varepsilon u(r)|^2 = \left|\int_r^q \frac{\mathrm{d}}{\mathrm{d}\theta}(P^\varepsilon u)\mathrm{d}\theta\right|^2 \le \mathrm{dist}(q,r)\frac{1}{\rho_e^*}\int_r^q \left|\frac{\mathrm{d}}{\mathrm{d}\theta}(P^\varepsilon u)\right|^2 \rho^* \mathrm{d}\theta$$

$$\le \mathrm{dist}(q,r)\frac{\varepsilon^{1-n}}{\rho_e^*}\frac{\alpha^{\overline{A}}}{\beta^A}\int_{V^\varepsilon} |\nabla u|^2 \mathrm{d}s\mathrm{d}\theta \le \mathrm{dist}(q,r)\frac{\varepsilon^{1-n}}{\rho_e^*}\frac{\alpha^{\overline{A}}}{\beta^A}\varepsilon^{n-1} \le C\mathrm{dist}(q,r)$$

for some constant C. Therefore, $|P^\varepsilon u(p_e) - P^\varepsilon u(p_{\tilde{e}})| \le 2C\sqrt{\mathrm{dist}(p_e, p_{\tilde{e}})}$. □

Theorem 7.2. *Let $u_\varepsilon \in H_0^1(T_N^\varepsilon)$ be an eigenfunction with eigenvalue λ_ε of the Laplace operator on T_N^ε, such that $\|u_\varepsilon\|_{L^2(T_N^\varepsilon)} = \varepsilon^{(N-1)/2}$. Assume that $\lim_{\varepsilon\to 0}\lambda_\varepsilon = \lambda^*$. Then there exists an eigenfunction u^* of L_* which corresponds to λ^*, such that up to a subsequence,*

$$u^* = \lim_{\varepsilon\to 0} P^\varepsilon u_\varepsilon$$

locally uniformly.

Proof. By elliptic regularity, $u_\varepsilon \in C^2(T_N^\varepsilon)$. Our proof consists of three steps.
Step 1. Let us show that $P^\varepsilon u_\varepsilon$ converges to a solution u^* of $\frac{d^2u}{d\theta^2} = \lambda^* u$ on each edge of $e \in T_1$.

By parts 2 and 4 of Lemma 6.4 (with $W = 1$), we obtain that $P^\varepsilon u_\varepsilon$ are uniformly bounded in $H_{*,1}(T_1)$. This implies, in particular, that $P^\varepsilon u_\varepsilon$ are uniformly locally bounded in $L^\infty(T_1)$. In addition, (up to a subsequence) $\lim_{\varepsilon\to 0} P^\varepsilon u_\varepsilon = u^*$ holds locally uniformly by Arzelà-Ascoli's Theorem. Fix an edge $e \in T_1$, and $\theta_1, \theta_2 \in e$. Let $\zeta(\theta) \in C_0^\infty([\theta_1,\theta_2])$. If $\varepsilon > 0$ is sufficiently small, then $\theta_1, \theta_2 \in \overline{E}^\varepsilon$. Therefore,

$$\begin{aligned}\int_{\theta_1}^{\theta_2} P^\varepsilon u_\varepsilon(\theta)\zeta''(\theta)d\theta &= \int_{\theta_1}^{\theta_2} \frac{1}{|\Omega^\varepsilon|}\left(\int_{\Omega^\varepsilon} u_\varepsilon(\theta,\mathbf{s})d\mathbf{s}\right)\zeta''(\theta)\,\mathrm{d}\theta \\ = \frac{1}{|\Omega^\varepsilon|}\int_{\Omega^\varepsilon}\int_{\theta_1}^{\theta_2} u_\varepsilon(\theta,\mathbf{s})\Delta\zeta(\theta)\,\mathrm{d}\theta\,\mathrm{d}\mathbf{s} &= -\frac{1}{|\Omega^\varepsilon|}\int_{\Omega^\varepsilon}\int_{\theta_1}^{\theta_2} \nabla u_\varepsilon(\theta,\mathbf{s})\cdot\nabla\zeta(\theta)\,\mathrm{d}\theta\,\mathrm{d}\mathbf{s} \\ = -\frac{\lambda_\varepsilon}{|\Omega^\varepsilon|}\int_{\Omega^\varepsilon}\int_{\theta_1}^{\theta_2} u_\varepsilon(\theta,\mathbf{s})\zeta(\theta)\,\mathrm{d}\theta\,\mathrm{d}\mathbf{s} &= -\lambda_\varepsilon\int_{\theta_1}^{\theta_2} P^\varepsilon u_\varepsilon(\theta)\zeta(\theta)\,\mathrm{d}\theta.\end{aligned}$$

Hence, $P^\varepsilon u_\varepsilon \in H^2([\theta_1,\theta_2])$ and $-(P^\varepsilon u_\varepsilon)'' = \lambda_\varepsilon P^\varepsilon u_\varepsilon$ in the weak sense and by elliptic regularity also in the strong sense. Moreover $P^\varepsilon u_\varepsilon$ is C^∞ in $\overline{E}^\varepsilon$. Since $\lambda_\varepsilon \to \lambda^*$ and $P^\varepsilon u_\varepsilon \to u^*$ uniformly on e, the second derivatives $(P^\varepsilon u_\varepsilon)''$ converge uniformly to $(u^*)''$, which also implies the same convergence for the first derivatives $(P^\varepsilon u_\varepsilon)'$.
Step 2. We show now that u^* is in the domain of L_*. For this, we must only show that u^* satisfies the corresponding Kirchhoff's conditions. The continuity at the vertices is satisfied by Lemma 7.1. The second Kirchhoff condition is given by

$$\sum_{e\in N(v)} \rho_e^* u_e'(v) = 0,$$

where $N(v)$ is the set of all edges adjacent to the vertex v. Recall that $\rho_e^* = \delta_e^{(N-1)}|\Omega_e|$ takes a constant value on each edge e.

Let $\overline{U} \subset T_1$ be a neighborhood of the vertex v which contains no other vertex, and let $\theta_e \in \partial\overline{U}$ be the point of $\partial\overline{U}$ contained in $e \in N(v)$. Let $U^\varepsilon \subset T_N^\varepsilon$ be the *inflation* of $\overline{U}$, that is, $\overline{U} = U^\varepsilon \cap T_1$. In particular, for sufficiently small ε we have

$$\partial U^\varepsilon = (U^\varepsilon \cap \partial T_N^\varepsilon)\bigcup_{e\in N(v)} S_e\,,$$

where $S_e = \{\mathbf{s};(\theta_e,\mathbf{s}) \in E^\varepsilon(e)\}$. Let $\zeta_\varepsilon \in C^\infty(U^\varepsilon)$ be a function which does not depend on $\mathbf{s}$ in the edges, satisfies $\zeta_\varepsilon(\mathbf{x}) = 1$ for all $x \in V^\varepsilon(v)$, $0 \le \zeta_\varepsilon(\mathbf{x}) \le 1$ for

all $x \in U^\varepsilon$, and vanishes around each S_e. Since u_ε is an eigenfunction, we have

$$\lambda_\varepsilon \int_{U^\varepsilon} u_\varepsilon \zeta_\varepsilon \,\mathrm{d}\mathbf{x} = \int_{V^\varepsilon(v)} \nabla u_\varepsilon \cdot \nabla \zeta_\varepsilon \,\mathrm{d}\mathbf{x} + \int_{U^\varepsilon(v)\setminus V^\varepsilon(v)} \nabla u_\varepsilon \cdot \nabla \zeta_\varepsilon \,\mathrm{d}\mathbf{x}$$
$$= \int_{U^\varepsilon(v)\setminus V^\varepsilon(v)} \frac{\partial u_\varepsilon}{\partial \theta}\frac{\mathrm{d}\zeta_\varepsilon}{\mathrm{d}\theta}\,\mathrm{d}\mathbf{s}\,\mathrm{d}\theta.$$

As ζ_ε depends only on θ on $U^\varepsilon(v)\setminus V^\varepsilon(v)$ and equals one at p_e, we get

$$\int_{U^\varepsilon(v)\setminus V^\varepsilon(v)} \frac{\partial u_\varepsilon}{\partial \theta}\frac{\mathrm{d}\zeta_\varepsilon}{\mathrm{d}\theta}\mathrm{d}\mathbf{s}\mathrm{d}\theta = \sum_{e\in N(v)} |\Omega_e^\varepsilon| \int_{p_e}^{\theta_e} \frac{\partial P^\varepsilon u_\varepsilon}{\partial \theta}\frac{\mathrm{d}\zeta_\varepsilon}{\mathrm{d}\theta}\,\mathrm{d}\theta \tag{7.2}$$
$$= -\sum_{e\in N(v)} |\Omega_e^\varepsilon| \left[\frac{\partial P^\varepsilon u_\varepsilon}{\partial \theta}(p_e)\zeta_\varepsilon(p_e) + \int_{p_e}^{\theta_e} (P^\varepsilon u_\varepsilon)''\zeta_\varepsilon\,\mathrm{d}\theta\right]$$
$$= -\sum_{e\in N(v)} |\Omega_e^\varepsilon| \left[\frac{\partial P^\varepsilon u_\varepsilon}{\partial \theta}(p_e) - \lambda_\varepsilon \int_{p_e}^{\theta_e} P^\varepsilon u_\varepsilon\zeta_\varepsilon\,\mathrm{d}\theta\right]$$
$$= -\sum_{e\in N(v)} |\Omega_e^\varepsilon| \frac{\partial P^\varepsilon u_\varepsilon}{\partial \theta}(p_e) + \lambda_\varepsilon \int_{U^\varepsilon(v)\setminus V^\varepsilon(v)} u_\varepsilon\zeta_\varepsilon\,\mathrm{d}\mathbf{x}.$$

The change of order of integration and differentiation in the first line of (7.2) is easily justified by approximating u_ε with a smooth function. We therefore obtain that

$$\sum_{e\in N(v)} |\Omega_e^\varepsilon| \frac{\partial P^\varepsilon u_\varepsilon}{\partial \theta}(p_e) = -\lambda_\varepsilon \int_{V^\varepsilon(v)} u_\varepsilon\zeta_\varepsilon\,\mathrm{d}\mathbf{x},$$

and since $|V^\varepsilon(v)| = c\varepsilon^N$, we arrive at the estimate

$$\left|\sum_{e\in N(v)} \rho_e^* \frac{\partial P^\varepsilon u_\varepsilon}{\partial \theta}(p_e)\right| \le c\varepsilon^{1-N}\varepsilon^{N/2}\lambda_\varepsilon \left(\int_{V^\varepsilon(v)} u_\varepsilon^2(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{1/2} \tag{7.3}$$
$$= c\lambda_\varepsilon\varepsilon^{1/2}.$$

Letting $\varepsilon \to 0$, we obtain by Step 1 that the left hand side of (7.3) converges to

$$\left|\sum_{e\in N(v)} \rho_e^*(u_e^*)'(v)\right|$$

and the right hand side to zero.

Step 3. It remains to prove that $u^* \not\equiv 0$. Let $T_{N,j}$ denote the j first generations in T_N. By lemmas 6.4 and 4.10 there are constants $c, C > 0$ and a function $R(j)$

which tends to zero as $j \to \infty$ such that

$$
\begin{aligned}
\varepsilon^{(N-1)} \int_{T_{1,j}} |P^\varepsilon u_\varepsilon|^2 \rho^* \mathrm{d}\theta
&= \varepsilon^{(N-1)} \int_{T_1} |P^\varepsilon u_\varepsilon|^2 \rho^* \mathrm{d}\theta - \varepsilon^{(N-1)} \int_{T_1 \setminus T_{1,j}} |P^\varepsilon u_\varepsilon|^2 \rho^* \mathrm{d}\theta \\
&\geq (1-\sqrt{\varepsilon})\varepsilon^{(N-1)} - c\lambda_\varepsilon \varepsilon^N - \frac{c^2}{C^2} R(j)^2 \lambda_\varepsilon \varepsilon^{(N-1)} \\
&= \varepsilon^{(N-1)} \left[(1-\sqrt{\varepsilon}) - c\lambda_\varepsilon \varepsilon - \frac{c^2}{C^2} R(j)^2 \lambda_\varepsilon \right].
\end{aligned}
$$

Choose $\varepsilon > 0$ small enough and j large enough so that $(1-\sqrt{\varepsilon}) - c\lambda_\varepsilon\varepsilon - \frac{c^2}{C^2}R(j)^2\lambda_\varepsilon \geq \gamma$ for a constant $\gamma > 0$. Then $\int_{T_{1,j}} |P^\varepsilon u_\varepsilon|^2 \rho^* \mathrm{d}\theta \geq \gamma > 0$. By the local uniform convergence of $P^\varepsilon u_\varepsilon$ to u^* we have that

$$
\int_{T_1} |u^*|^2 \rho^* \, \mathrm{d}\theta \geq \int_{T_{1,j}} |u^*|^2 \rho^* \, \mathrm{d}\theta = \lim_{\varepsilon \to 0} \int_{T_{1,j}} |P^\varepsilon u_\varepsilon|^2 \rho^* \, \mathrm{d}\theta \geq \gamma > 0,
$$

so, $u^* \not\equiv 0$. $\square$

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